

An Analysis of Cascade Chains in the Stern-Brocot Sequence

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Abstract

We introduce and analyze the algebraic framework of finite cascade sequences, defined as deterministic paths terminating in a unique absorbing state [1][2]. Three complementary perspectives are developed: graph-theoretic, probabilistic, and operator-theoretic. In the graph view, cascade sequences form directed acyclic graphs with a terminal sink [3][4]. In the Markov chain formulation, they correspond to absorbing chains with deterministic absorption time [7][8][13]. In the operator view, the transition operator T is shown to be nilpotent, generating a finite algebra with minimal polynomial x^k [9][15].

Further the semi group $\{T, T^2, \dots, T^k\}$ under composition, the monoid obtained by adjoining the identity and the absorbing element T^k are characterized [5][6][11][12]. These are shown to be isomorphic to truncated polynomial semigroups and monoids. Extending the algebraic cascade property, we prove that each finite cascade sequence is a cyclic module over the truncated polynomial ring $Q[x]/(x^k)$, with dimension k over Q [9].

Together, these results unify combinatorial, probabilistic, and algebraic aspects of cascade sequences establishing a rigorous foundation for their study and connecting them to classical structures in semigroup theory, ring theory, and module theory [2][5][6].

1. Introduction:

Cascade Chains arise as deterministic processes that evolve through a finite chain of states before terminating in a unique absorbing element [1][3]. Their simplicity belies a rich algebraic structure, connecting ideas from graph theory, probability, and operator algebras [2][4][7]. In this paper, we develop a unified framework for cascade sequences, establishing their fundamental properties and situating them within classical mathematical structures [5][6][9].

From a graph-theoretic perspective, cascade sequences correspond to directed paths ending in a sink vertex [3][14]. In the probabilistic setting, they can be viewed as absorbing Markov chains with deterministic absorption time [7][8][13]. From the operator-theoretic viewpoint, the transition operator T is nilpotent,

generating a finite algebra with minimal polynomial x^k [9][15]. These perspectives converge to reveal algebraic cascade property.

Building on this foundation, we investigate semigroup and monoid structures arising from powers of T , and show that they are isomorphic to truncated polynomial semigroups and monoids. We further extend the operator algebra to a module-theoretic setting, proving that cascade sequences form cyclic modules over the truncated polynomial ring $Q[x]/(x^k)$ with finite dimension[9]. This situates cascade sequences within the broader context of representation theory and finite-dimensional algebras [15].

The results presented here unify combinatorial, probabilistic, and algebraic

aspects of cascade sequences, providing a rigorous foundation for their study and opening pathways to further exploration in semigroup theory, ring theory, and module theory[2][5][6].

2. Chain Transformations

2.1. Definition (Chain Transformation Rule):

Let $\frac{p}{q}$ be a fraction in $[0,1]$ appearing at level i of the Stern-Brocot sequence. The cascade transformation rule is defined by

$$\frac{p}{q} = 1 - \frac{q-p}{q} = 1 - \frac{1}{q/(q-p)} = 1 - \frac{1}{1+\frac{r}{s}}$$

where $\frac{r}{s}$ denotes the next fraction in the chain. This transformation initiates the chain associated with $\frac{p}{q}$, and subsequent steps are obtained by recursively applying the same rule.

2.2 Theorem (Termination of Chain):

Let $i \geq 2$ be a fixed level of the Stern-Brocot sequence in $[0,1]$. For every fraction $\frac{p}{q}$ appearing at level i , the chain generated by the transformation rule is finite and terminates at $\frac{0}{1}$. Moreover, the penultimate term of each chain is always one of the fractions

$$\frac{i-1}{i}, \frac{i-2}{i-1}, \frac{i-3}{i-2}, \dots, \frac{1}{2}.$$

Proof:

At level i of the Stern-Brocot sequence, the number of fractions is finite, namely $2^{i-1} + 1$. Therefore, the number of cascade chains that can be generated from this level is also finite.

Consider a fraction $\frac{p}{q}$ at level i . By the transformation rule, each step rewrites $\frac{p}{q}$ into a new fraction with strictly smaller numerator. Since the numerator is a nonnegative integer, this process must

eventually reach zero, yielding the absorbing fraction $\frac{0}{1}$. Thus every cascade chain terminates at $\frac{0}{1}$.

Furthermore, We know that every chain terminates at $\frac{0}{1}$. By using transformation

rule in the reversal form we get $\frac{0}{1} = 1 - \left(\frac{1}{i+\frac{0}{1}}\right) = 1 - \frac{1}{i} = 1 - \frac{1}{i} = \frac{i-1}{i}$. If we put

$i = i - 1$ then we get $\frac{i-2}{i-1}$. From the above we get the penultimate term for each Cascade Chain is one of the following, $\frac{i-1}{i}, \frac{i-2}{i-1}, \frac{i-3}{i-2}, \dots, \frac{1}{2}$.

3. Graphical Representation

3.1 Definition (Graph Representation of Chains):

Given a collection of chains $C_1, C_2, \dots, C_n$. We construct a directed graph G as follows:

- Vertices: all fractions appearing in the chains, together with the absorbing vertex $\frac{0}{1}$.

- Edges: for each chain $C_j = \{a_1, a_2, \dots, a_k, \frac{0}{1}\}$, include directed edges

$$(a_1, a_2), (a_2, a_3), \dots, \left(a_k, \frac{0}{1}\right).$$

- Absorbing sink: the vertex $\frac{0}{1}$ has no outgoing edges.

3.2 Theorem: Let $S_1, S_2, \dots, S_n$ be finite

sequences of rational numbers where each sequence terminates at 0. Construct a directed graph G whose vertices are the fractions in all sequences together with the node 0, and whose edges represent transitions between consecutive terms. Then G is a directed acyclic graph. Each sequence corresponds to a directed path ending at 0. The node 0 has in-degree n and out-degree 0 and every other node has out-degree exactly 1.

Proof: Consider the construction of G . Each sequence $S_j = \{a_1, a_2, \dots, a_k, 0\}$ contributes edges $(a_1, a_2), (a_2, a_3), \dots, (a_k, 0)$. Since the sequences are finite and terminate at 0, no sequence can revisit a previous fraction, and hence no directed cycle can occur. Therefore G is acyclic.

For each sequence, the edges form a directed path beginning at its first fraction and ending at 0. Thus every sequence corresponds to a directed path terminating at the absorbing node. Since each sequence ends at 0, there is exactly one edge entering 0 from each sequence. So the in-degree of 0 is n . The node 0 has no successors. So its out-degree is 0.

Since, every non-terminal fraction a_i has exactly one successor a_{i+1} in its sequence, each non-terminal node has out-degree equal to 1. We have established that G is a directed forest of paths converging into the single absorbing sink at 0.

4. Markov Chain Representation

4.1 Definition (Markov Chain Representation Chains): Given a collection of chains $C_1, C_2, \dots, C_n$. We define a Markov chain M with state space equal to all fractions appearing in the chains together with the absorbing state $\frac{0}{1}$.

The transition probabilities are deterministic: $P(a_i, a_{i+1}) = 1$, "for each consecutive pair in a chain" $P\left(\frac{0}{1}, \frac{0}{1}\right) = 1$.

Thus, every non-terminal fraction transitions with probability 1 to its successor, while the absorbing state $\frac{0}{1}$ remains fixed. This construction yields an absorbing Markov chain with a unique absorbing state at $\frac{0}{1}$.

4.2 Definition (Transition Operator Representation of Chains): Let $C_1, C_2, \dots, C_n$ be chains with terminal state

$\frac{0}{1}$. Define a transition operator T acting on the state space of all fractions together with $\frac{0}{1}$ by $T(a_i) = a_{i+1}$, $T\left(\frac{0}{1}\right) = \frac{0}{1}$. Repeated application of T eventually yields $\frac{0}{1}$ for every initial fraction. Thus, T is nilpotent on transient states and $\frac{0}{1}$ is the unique absorbing fixed point.

4.3 Theorem: Let M be the Markov chain defined on a finite cascade sequence $S = \{a_1, a_2, \dots, a_k, 0\}$ with transition operator T given by $T(a_i) = a_{i+1}$ and $T(0) = 0$. Then:

1. M is absorbing with unique absorbing state 0.
2. The transition matrix P admits block form $P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$
3. The fundamental matrix $N = (I - Q)^{-1}$ exists.
4. The absorption time equals k .

Proof: Since $T(0) = 0$, the chain reaches 0, establishing that 0 is absorbing state. Each non-terminal state a_i transitions deterministically to its successor a_{i+1} . So the transition matrix has the stated block form. Because the sequence is finite, repeated application of T eventually maps every transient state to 0. This implies that Q is nilpotent, hence $I - Q$ is invertible and the fundamental matrix $N = (I - Q)^{-1}$ exists. Finally, starting at a_1 , the chain moves deterministically through $a_2, a_3, \dots, a_k$ before reaching 0. So absorption occurs after exactly k steps. This completes the proof.

5. Algebraic Property

5.1 Theorem: Let T be the transition operator defined on a finite sequence

$S=\{a_1, a_2, \dots, a_k, 0\}$ where $T(a_i)=a_{i+1}$ and $T(0)=0$. Then:

1. T is nilpotent on transient states.
2. The algebra generated by T is finite and contains the absorbing element 0.
3. The minimal polynomial of T is x^k , where k is the maximum sequence length[5][6][11][12].

Proof: For any transient state a_i , repeated application of T yields $T(a_i)=a_{i+1}, T^2(a_i)=a_{i+2}, \dots$. After k steps we obtain $T^k(a_i)=0$. Thus T is nilpotent on transient states.

Since T acts on a finite set of states and $T^k=0$, the algebra generated by $\{I, T, T^2, \dots, T^{k-1}\}$ is finite and necessarily contains the absorbing element 0. Finally, because $T^k=0$ but $T^{k-1} \neq 0$ (for example, $T^{k-1}(a_1)=a_k$), the smallest polynomial that annihilates T is x^k . Therefore, the minimal polynomial of T is x^k .

5.1.1 Corollary: The algebra generated by the transition operator T is isomorphic to the truncated polynomial ring $Q[x]/(x^k)$, where k is the maximum sequence length[14].

Proof: From previous Theorem, we know that $T^k=0$ but $T^{k-1} \neq 0$. The algebra generated by $\{I, T, T^2, \dots, T^{k-1}\}$ is closed under addition and multiplication, and T^k vanishes. The relations among powers of T are exactly those of polynomials modulo x^k , establishing the isomorphism.

5.2 Theorem: Let T be the transition operator defined by $T(a_i)=a_{i+1}$, $T(a_0)=0$ on a finite sequence $S=\{a_1, a_2, \dots, a_n, a_0\}$. Then the set $\{T, T^2, T^3, \dots, T^n\}$ under composition forms a semigroup [5][6].

Proof: For any $m, n \geq 1$, we have $(T^m \circ T^n)(a_i) = T^{m+n}(a_i)$. Thus closure under composition is satisfied, and associativity follows from the general property of function composition. Therefore, the set

$\{T, T^2, \dots, T^n\}$ forms a semigroup under composition.

5.3 Theorem: If we adjoin the identity operator I defined by $I(a_i)=a_i$ for all states, then the set $\{I, T, T^2, \dots, T^n\}$ under composition forms a monoid[14].

Proof: From the previous theorem, $\{T, T^2, \dots, T^n\}$ is a semigroup. Adding I gives $I \circ T^m = T^m \circ I = T^m$ for all m . Thus I acts as the identity element, so the set $\{I, T, T^2, \dots, T^n\}$ is a monoid under composition.

5.4 Proposition: The operator T^k acts as a zero element in both the semigroup and monoid structures[11][12].

Proof: By definition, T^k maps every state to 0. For any $m \geq 1$, composing T^m with T^k yields

$(T^m \circ T^k)(a_i) = T^k(a_i)$ since once a state is mapped to 0, further applications of T leave it at 0. Similarly, $(T^k \circ T^m)(a_i) = T^k(a_i)$. Thus T^k is absorbing in both the semigroup $\{T, T^2, \dots, T^n\}$ and the monoid $\{I, T, T^2, \dots, T^n\}$.

5.5 Corollaries on Semigroup and Monoid Structures

5.5.1 Corollary: The semigroup $\{T, T^2, T^3, \dots, T^k\}$ under composition is isomorphic to the semigroup of polynomials $\{x, x^2, x^3, \dots, x^k\}$ under multiplication modulo x^k .

Proof: Define a mapping $\varphi(T^m)=x^m$ for $1 \leq m \leq k$. Composition of operators corresponds to multiplication of polynomials: $\varphi(T^m \circ T^n) = \varphi(T^{m+n}) = x^{m+n}$. Since T^k acts as zero, this corresponds to $x^k=0$ in the truncated polynomial semigroup. Thus φ is an isomorphism.

5.5.2 Corollary: The monoid $\{I, T, T^2, T^3, \dots, T^k\}$ is isomorphic to the truncated polynomial monoid

$\{1, x, x^2, x^3, \dots, x^k\}$ under multiplication modulo x^k .

Proof: Extend the mapping φ by setting $\varphi(I)=1$. Then $\varphi(T^m)=x^m$ for $m \geq 1$. The identity operator I corresponds to the multiplicative identity 1, and the absorbing element T^k corresponds to $x^k=0$. Therefore, the monoid structure is preserved under φ .

6. Cascade Sequence as a Cyclic Module

6.1 Theorem: Let $S=\{a_1, a_2, \dots, a_k, 0\}$ be a finite cascade sequence with transition operator T defined by $T(a_i)=a_{i+1}$ and $T(0)=0$. Then S can be regarded as a cyclic module over the truncated polynomial ring $\mathbb{Q}[x]/(x^k)$, generated by a_1 , with module action $x^m \cdot a_1 = a_{m+1}$, $1 \leq m \leq k$, $a_{k+1}=0$.

Proof: From the algebraic cascade property, the operator algebra generated by T is isomorphic to $\mathbb{Q}[x]/(x^k)$, with T corresponding to x . Define the module action by $x^m \cdot a_1 = T^m(a_1) = a_{m+1}$. Since $T^k(a_1)=0$, this corresponds to $x^k=0$ in the truncated polynomial ring. Thus the cascade sequence S is a cyclic module over $\mathbb{Q}[x]/(x^k)$, generated by a_1 .

6.1.1 Corollary: The cascade sequence $S=\{a_1, a_2, \dots, a_k, 0\}$, regarded as a cyclic module over $\mathbb{Q}[x]/(x^k)$, has dimension k as a vector space over $\mathbb{Q}$.

Proof: The module is generated by a_1 , with action $x^m \cdot a_1 = a_{m+1}$ for $1 \leq m \leq k$. Thus the set $\{a_1, a_2, \dots, a_k\}$ forms a spanning set. Moreover, these elements are linearly independent over $\mathbb{Q}$, since each a_i corresponds to a distinct state in the cascade sequence. Therefore, the module has dimension k as a vector space over $\mathbb{Q}$.

7. Applications:

- **Automata Theory:** Cascade sequences model deterministic finite automata with absorbing states [4].

- **Markov Chains:** They correspond to absorbing chains with deterministic absorption time, aligning with classical treatments of absorbing Markov Chains[7][8][13].
- **Semigroup and Ring Theory:** Provide examples of truncated polynomial semigroups, monoids, and cyclic modules [14].
- **Graph Theory:** Represent directed paths ending in sinks, useful in network flow and cascade failure analysis [3].
- **Computer Science:** Model recursive processes with bounded depth and finite termination [1][2].
- **Applied Domains:** Applications in epidemiology (finite infection chains), economics (bankruptcy models), and engineering (reliability chains)[13].

8. Conclusion:

Cascade sequences provide a unified framework that connects graph theory, probability, and algebra through the study of deterministic processes with absorbing states [1][3][4]. By analyzing the transition operator T , we established its nilpotent nature[9] and demonstrated how its powers generate finite semigroup and monoid structures [5][6][11][12]. These were shown to be isomorphic to truncated polynomial semigroups and monoids, revealing deep algebraic parallels. Extending further, we proved that cascade sequences form cyclic modules over the truncated polynomial ring $\mathbb{Q}[x]/(x^k)$, with dimension k , situating them naturally within representation theory [9][15]. Together, these results highlight the versatility of cascade sequences, bridging combinatorial and algebraic viewpoints while opening pathways to applications in automata theory[4], Markov chains[7][8][13], network analysis[3], and

applied modeling in reliability and systems theory[13]. This work lays a rigorous foundation for future exploration of cascade sequences across both pure and applied mathematics[2][5][6].

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