The Bioscan — AN INTERNATIONAL QUARTERLY JOURNAL OF LIFE SCIENCES

21(3): 41-51, 2026

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ANALYTIC ASPECTS OF PARTITION AND COLORED PARTITION FUNCTIONS USING q-SERIES

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DOI:https://doi.org/10.63001/tbs.2026.v21.i03.pp41-51

KEYWORDS
Weighted partitions, linear recurrence relations, asymptotic analysis, three-dimensional visualization, divisor sums, computational number theory.

Received on: 04-05-2026
Accepted on: 04-06-2026
Published on: 02-07-2026

Abstract

A weighted partition generating function defined by a multi-factor Euler product using analytic and computational methods. Explicit coefficient formulas are derived via convolutions of colored partition functions and Euler-type linear recurrence relations involving divisor sums are established. Asymptotic analysis shows that the coefficients exhibit exponential growth governed by dominant singularities on the boundary of the unit disk. Numerical evaluation based on truncated Euler products is investigated, and three-dimensional surface plots confirm stable convergence within the unit disk while indicating increased sensitivity near the dominant singularity. The combined results highlight the usefulness of integrating theoretical analysis with numerical computation for partition-type generating functions.

1. Introduction

Partition theory provides a powerful framework linking additive number theory, enumerative combinatorics, and several models in mathematical physics, largely through the use of generating functions and q-series expansions. Let $P(m)$, denote the classical partition function, defined as the number of unordered decompositions of a positive integer mmm into a sum of positive integers. Its ordinary generating function admits the Euler product representation

$$P(q) = \sum_{m=0}^{\infty} P(m)q^m = \prod_{m=1}^{\infty} (1 - q^m)^{-1} \qquad (1)$$

The behaviour of $P(q)$ as a complex function is fundamental in the analytic theory of partitions. In particular, the function is holomorphic in the unit disk and develops its dominant singular behaviour as q approaches the unit circle, which governs the rapid growth of the coefficients P(m). The classical circle method of Hardy and Ramanujan established the first precise asymptotic description of P(m) and provided a general approach for partition-type sequences [1]. This line of research was later sharpened through Rademacher's exact convergent series and further developments connecting partition generating functions with modular forms and complex analytic techniques [2, 3, 4, 10].

Since both the series and the Euler product in (1.1) converge uniformly on compact subsets of $\{Q \in C: |q| < 1\}$, differentiation with respect to q is justified and may be carried out termwise [3, 5]. Such differentiation naturally produces new generating functions encoding weighted partition statistics and refined invariants, including those related to rank and crank phenomena [3, 4, 6]. More generally, analytic control of these generating functions supports the derivation of identities, recurrence relations, and asymptotic expansions by combining tools from q-series, modular forms, and complex analysis [2, 6, 10]. Analogous analytic considerations also appear in statistical mechanics, where partition functions represent thermodynamic quantities: smooth dependence on parameters corresponds to analyticity, while singularities are closely associated with critical behaviour and phase transitions [11, 12].

In this work, we focus on the analyticity and differentiability of partition and colored partition generating functions. We identify conditions under which these functions possess derivatives of arbitrary order and examine the consequences of these properties for partition identities and structural results, thereby enriching the analytic framework of partition theory.

Colored partition functions extend the classical theory of partitions by permitting each part to appear in a fixed number of distinct colors. Let $P_r(m)$ denote the number of partitions of a positive integer m in which every part is available in r colors [5]. The corresponding generating function is

$$P_r(q) = \sum_{m=0}^{\infty} P_r(m) q^m = \prod_{m=1}^{\infty} (1 - q^m)^{-r} \tag{2}$$

From the standpoint of complex analysis, the above infinite product converges absolutely and uniformly on compact subsets of the open unit disk |q| < 1. As a result, $P_r(q)$ is analytic in this region and possesses derivatives of all orders with respect to q. This property legitimizes term wise differentiation of the generating function and leads to new series whose coefficients represent weighted and refined colored partition functions.

The differentiability of $P_r(q)$ is a key tool in establishing identities, recurrence relations, and structural properties of colored partitions. Furthermore, the analytic behavior of these generating functions allows the effective application of techniques from q-series, modular forms, and analytic number theory. Therefore, examining the analyticity and differentiability of colored partition functions enriches the theoretical foundation of partition theory and opens avenues for further generalizations.

1.1. Continuity of Partition and Colored Partition Functions:

Continuity is one of the basic analytic properties required for a rigorous study of partition generating functions and their extensions. For both the classical partition function and colored partition functions, continuity follows directly from the convergence behavior of the associated power series and infinite products.

1.2. Continuity of the Classical Partition Generating Function:

The generating function of ordinary partition function $P(m)$ is converges absolutely and uniformly on closed disks contained in the unit disk. Since each partial sum is a continuous function of q, uniform convergence implies that the limiting function $P(q)$ is continuous in $|q| < 1$. An equivalent conclusion is obtained from the infinite product representation, as each factor $(1 - q_n)^{-1}$ is continuous and the product converges uniformly on compact subsets of the domain.

Let $Q \subset \mathrm{C}$ and let $h: Q \to C$ be a complex-valued function.

The function h is said to be continuous at a point $q_0 \in Q$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|q - q_0| < \delta \Rightarrow |h(q) - h(q_0)| < \varepsilon$.

If h is continuous at every point of Q, then h is said to be continuous on Q.

Equivalently, a function is continuous at q_0 if

$$\lim_{q \to q_0} h(q) = h(q_0)$$

1.3. Continuity of Colored Partition Generating Function:

Let $r \in N$ and let $P_r(m)$ denote the number of r-colored partitions of a nonnegative integer m.

When viewed as a function of the complex variable q, the generating function $P_r(q)$is continuous in the open unit disk $Q = \{q \in C: |q| < 1\}$. This follows from the fact that, for any fixed

$s < 1$, the infinite product converges uniformly on the closed disk$|q| \leq s$. Each factor $(1 - q_n)^{-r}$ is continuous for $|q| < 1|$, and uniform convergence of the partial products guarantees continuity of the limit function.

Continuity generally fails on the boundary $|q| = 1$ because the factors $(1 - q_n)^{-r}$develop singularities as q approaches roots of unity. Therefore, $P_r(q)$ is continuous on every compact subset of Q, but not extendable as a continuous function to the entire unit circle.

The continuity of both $P(q)$ and $P_r(q)$ provides the necessary analytic groundwork for further investigations. It justifies subsequent operations such as differentiation and analytic continuation and supports the application of methods from complex analysis, q-series, and modular form theory. Moreover, continuity is essential for studying limiting behavior as q approaches the boundary of the unit disk, which is closely linked to asymptotic properties of partition functions.

2. r-Colored Partition Function of an integer m with same number of parts

The r-color Partition of an integer m is the combination of copies of integer m with respect to the corresponding color r. To illustrate the effect of coloring on the growth of partition functions, we present in table a numerical comparison of the ordinary partition function $P(m)$ and the r-colored partition functions $P_r(m)$ for $1 \le r \le 10$ and $0 \le m \le 15$. The table serves as a computational complement to the analytic results discussed earlier and offers insight into the dependence of partition growth on the number of colors.

m	P(m)	P_2(m)	P_3(m)	P_4(m)	P_5(m)	P_6(m)	P_7(m)	P_8(m)	P_9(m)	P_{10}(m)
0	1	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9	10
2	2	5	9	14	20	27	35	44	54	65
3	3	10	22	40	65	98	140	192	255	330
4	5	20	51	105	190	315	490	735	1050	1430
5	7	36	108	252	506	918	1540	2400	3555	5512
6	11	65	221	574	1265	2499	4424	7296	11440	19415
7	15	110	429	1240	2975	6350	11704	20544	33915	63570
8	22	185	810	2565	6630	15295	30240	56880	100440	195910
9	30	300	1479	5140	14190	35028	73815	147840	277920	573430
10	42	481	2640	10010	29393	77220	173745	369512	738480	1605340
11	56	752	4599	19096	58905	164472	393120	885720	1860480	4322110
12	77	1165	7868	35640	114730	339339	859320	2042040	4502040	11240645
13	101	1770	13209	64980	217800	697620	1860320	4659200	10740480	28341730
14	135	2665	21933	116280	402675	1428570	4008645	10400640	25045020	69488650
15	176	3956	35868	204204	729300	2917200	8593200	23279256	58198140	166096270

Table:1.1

The data illustrate the rapid increase in the number of partitions as the number of available colors grows. For fixed m, the sequence $P_r(m)$ is strictly increasing in r, reflecting the additional combinatorial freedom introduced by coloring. Even for small values of m, colored partition numbers dominate the ordinary partition function, highlighting their faster growth rate. This behavior is consistent with the known asymptotic formula for $P_r(m)$, in which the exponential growth term depends explicitly on the number of colors r. This behavior is consistent with the exponential growth predicted by the asymptotic relation

$$P_r(m) \approx e^{\left(\pi\sqrt{\frac{2rm}{3}}\right)}, \qquad m \to \infty.$$

If $P_r(m)$ be r-colored partition function of an integer m, then asymptotic relation of the partition generating function is

$$P_r(m) = \frac{1}{(2\pi)^{\frac{(r+1)}{2}}} m^{\frac{-(m+3)}{4}} e^{\left(\pi\sqrt{\frac{2rm}{3}}\right)}, \qquad m \to \infty, \tag{3}$$

where r and m are positive integers.

This function captures asymptotic behavior commonly observed in combinatorial sequences and partition-type problems.

The logarithmic representation of this function is

$$logP_r(m) = -\frac{(r+1)}{2}\log(2\pi) - \frac{(m+3)}{4}\log m + \pi\sqrt{\frac{2rm}{3}} \tag{4}$$

Due to the combination of the algebraic decay term $m^{\frac{-(m+3)}{4}}$ and the rapidly growing exponential $e^{\left(\pi\sqrt{\frac{2rm}{3}}\right)}$ a graphical representation is particularly useful to visualize growth trends across different values of r and m, highlight asymptotic behavior and the dominance of the exponential term for large mmm and understand surface behavior in a 3D context by plotting $P_r(m)$ against m and r.

3. Results and discussion

To better understand the behaviour of the function $P_r(m)$, we generated 3D surface plots illustrating both $P_r(m)$ and its logarithm $logP_r(m)$. The plot of $P_r(m)$ clearly shows the rapid exponential growth of the function as m increases, while the logarithmic plot highlights the asymptotic trends and dependence on the parameter r in a more interpretable manner. These visualizations provide an intuitive representation of the function's growth and the interplay between the algebraic decay and exponential terms, which are difficult to discern from the formula alone.

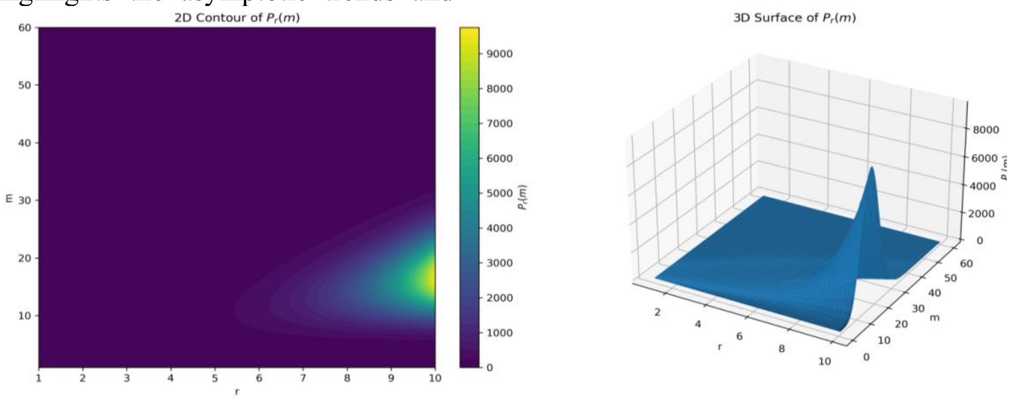


Fig. 1: plot of the function $P_r(m)$

Combined visualization of the function $\mathrm{P_r(m)}$, the left panel shows the filled contour plot of $\mathrm{P_r(m)}$over the parameter domain, while the right panel presents the corresponding 3D surface. The surface is characterized by a sharply localized peak, reflecting the rapid variation of $\mathrm{P_r(m)}$ across the $(\mathrm{r, m})$ - plane.

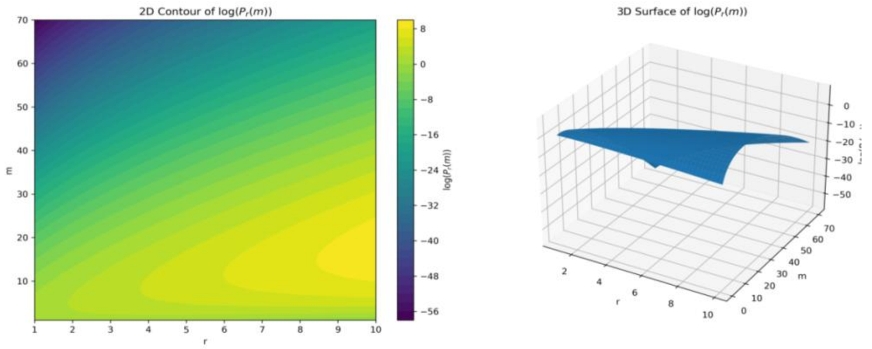


Fig. 2: plot of the function $logP_r(m)$

Since the function $P_r(m)$ varies across several orders of magnitude as r and m change, its direct 3D surface plot is dominated by a narrow peak, while most of the remaining region appears nearly flat and close to zero. To obtain a more informative global visualization, this transformation preserves the relative ordering of values and the location of extrema because the logarithm is strictly monotone, while simultaneously compressing large values and expanding small values. As a result, the plot of reveals the underlying geometric structure more clearly, including smooth gradients and ridge-like regions that are not visible in the original scale. Moreover, taking logarithms yields the explicit decomposition $\log P_r(m)$, which makes the competing growth and decay mechanisms transparent and supports asymptotic and numerical interpretation.

The linear plot of $P_r(m)$ shows the rapid exponential growth, which quickly becomes numerically large, while the logarithmic plot $\log P_r(m)$ clearly reveals the underlying asymptotic trends and parameter dependence. Comparing both plots highlights how logarithmic scaling provides an intuitive view of growth behavior and makes the function's structure easier to analyze. The figures were created using Python for numerical computation and plotting.

Theorem 3.1

If $P_r(m)$ is r-colored partition of an integer m, let $\{P_r(m)\}_{m\geq 0}$ be a sequence for r colored partition of m and who's ordinary generating function has the form

$$\delta(y) = \sum_{m\geq 0} P_r(m)\, y^m = \frac{\vartheta(y)}{(1-y)^k \rho(y)} \quad (5)$$

where $\vartheta(y)$ is a polynomial, $\mathrm{k} \geq 0$ is an integer and

$$\rho(y) = 1 - a_1 y - a_2 y^2 - \cdots - a_t y^t, \quad \rho(0) = 1. \quad (6)$$

is a polynomial with $\rho(0) = 1$.
The recursive formula of the sequence is

$$P_r(m) = \sum_{i=1}^{t} a_i P_r(m-i) + p_{k-1}(m), \quad m \geq t, \quad (7)$$

where $p_{k-1}(m)$ is a polynomial in m of degree at most $k-1$.

Lemma 3.1

Let

$$A(y) = \sum_{m\geq 0} f_m\, y^m = \frac{\vartheta(y)}{\rho(y)}, \quad (8)$$

where $\vartheta(y)$ is a polynomial and $\rho(y)$ is given by (2.2).
Then the coefficients $\{f_m\}$ satisfy

$$f_m = \sum_{i=1}^{t} a_i f_{m-i}, \quad m \geq t \quad (9)$$

Proof:
From (8) we have

$$A(y)\rho(y) = \vartheta(y). \quad (10)$$

Expanding $\rho(y)$ gives

$$(1 - a_1 y - a_2 y^2 - \cdots - a_t y^t) \sum_{m\geq 0} f_m\, y^m = \vartheta(y). \quad (11)$$

Equating coefficients of y^m yields

$$f_m - \sum_{i=1}^{\min(t,m)} a_i f_{m-i} = [y^m]\vartheta(y) \quad (12)$$

Since $\vartheta(y)$ is a polynomial, the coefficient $[y^m]\vartheta(y)$ vanishes for all sufficiently large m. In particular, for all $m \geq t$ beyond deg (ϑ), equation (2.8) reduces to

$$f_m = \sum_{i=1}^{t} a_i f_{m-i}.$$

Hence (9) holds.

□

Proof of Theorem 3.1
Multiply both sides by $(1-y)^k \rho(y)$ in (5)

$$(1-y)^k \rho(y)\delta(y) = \vartheta(y) \quad (13)$$

the auxiliary power series of (12) is

$$P(y) = (1-y)^k \rho(y) = \sum_{m\geq 0} p_m y^m \quad (14)$$

Then (13) becomes

$$P(y)\delta(y) = \vartheta(y) \quad (15)$$

By Lemma 2.1, the coefficients $\{p_m\}$ satisfy the homogeneous recurrence

$$p_m = \sum_{i=1}^{t} a_i p_{m-i}, \quad m \geq t \quad (16)$$

Since

$$(1-y)^k = \sum_{j=1}^{k} (-1)^j \binom{k}{j} y^j, \quad (17)$$

Substituting (17) into (15) and isolating $P_r(m)$ gives a linear recurrence for $P_r(m)$.
Since the coefficients of $(1-y)^k$are binomial polynomials in mmm, this

additional contribution can be written as a polynomial $p_{k-1}(m)$ with

$$\deg(p_{k-1}) \leq k-1.$$

Therefore, for all $m \geq t$ the recurrence (2.2) holds.

□

When $k=0$ in (5), the recurrence reduces to a homogeneous linear recurrence with polynomial coefficients. For $k \geq 1$, the presence of a polynomial forcing term of degree k makes the recurrence non-homogeneous.

Applications

1. If k = 2 and $\rho(y) = 1 - y - y^2$ in (5), this becomes $\delta_1(y) = \frac{1-y}{(1-y)^2(1-y-y^2)}$ and (6) becomes $c_m = 2c_{m-1} - c_{m-2} + F_{m-1}$, where F_{m-1} is Fibonacci sequence. The Sequence corresponds to $k=2$ and $\rho(y) = 1 - y - y^2$, yielding Fibonacci-type exponential growth.
2. If $k=1$ and $\rho(y) = 1 - 2y - y^2 + y^3$ in (5), this becomes $\delta_2(\mathrm{y}) = \frac{1+2\mathrm{y}+4\mathrm{y}^2}{(1-\mathrm{y})^2(1-2\mathrm{y}-\mathrm{y}^2+\mathrm{y}^3)}$ and (6) becomes $c_m = 2c_{m-1} + c_{m-2} - c_{m-3} + 1$. The Sequence corresponds to $k=1$ and $\rho(y) = 1 - 2y - y^2 + y^3$, yielding exponential growth with a different base.
3. If $k=6$ and $\rho(y) = 1$ in (5), this becomes $\delta_1(y) = \frac{1+y+y^2}{(1-y)^6}$ also (6) becomes $c_m = \binom{m+5}{5} + O(m^4)$. The Sequence corresponds to $k=6$ and $\rho(y) = 1$, yielding pure polynomial growth of order 4.

The mathematical description of the results illustrated by the Table 1.1 and expressed in terms of relations among the values of $P_r(m)$.

3.1 Monotonicity in the Number of Colors

For each fixed $m \geq 1$, the colored partition numbers increase with the number of colors r,

$$P_1(m) < P_2(m) < P_3(m) < \cdots < P_r(m)$$

or more generally, for any $r_1 < r_2$,

$$P_{r_1}(m) < P_{r_2}(m)$$

This follows directly from the generating function identity $P_r(q) = (P_1(q))^r$, since increasing r corresponds to taking higher powers of the series with non-negative coefficients.

3.2 Growth with Respect to m

For fixed r, the sequence $\{P_r(m)\}$ is strictly increasing: $P_r(m) < P_r(m+1),\ m \geq 1,$ which reflects the combinatorial fact that partitions of $m+1$ can be constructed by adding a part to partitions of m.

3.3 Convolution Formula

The table values satisfy the inequalities $P(m) = P_1(m) < P_r(m)$ for all $r > 1$, and more generally, by the binomial expansion of generating functions,

$$P_r(m) = \sum_{m_1+m_2+\ldots+m_r=m} P(m_1)P(m_2)\ldots P(m_r),\ r \geq 1$$

where the sum is over all non-negative integers $m_1, m_2, \ldots m_r$ summing to m. This explicitly shows that $P_r(m)$ is a convolution of r ordinary partition sequences.

4. Restricted 2-Colored Partition Function with Triple function and its Analyticity.

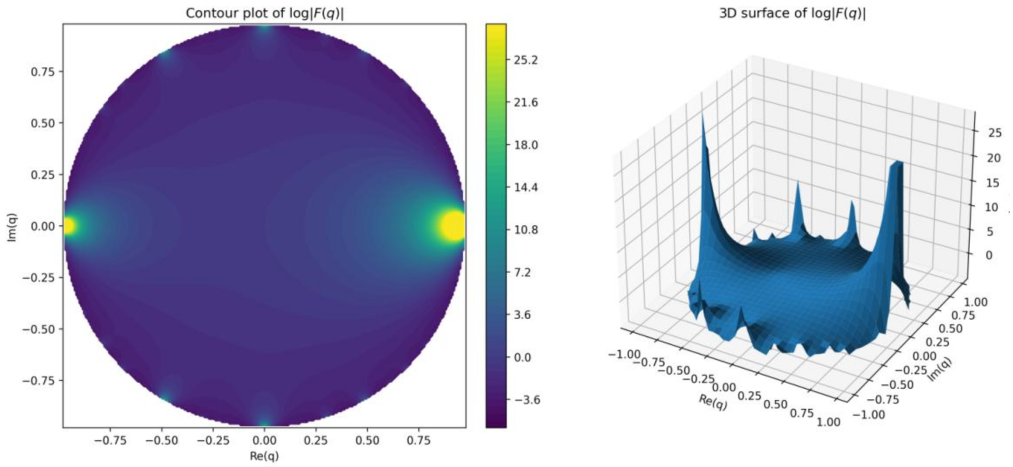
Infinite q-products play a central role in combinatorics and number theory, capturing structures such as integer partitions, color-constrained partitions, and sequences with arithmetic restrictions. These generating functions reveal deep links between enumeration and the analytic properties of series, forming the foundation for MacMahon-type q-series, Rogers–Ramanujan identities, and related results.

In this session, we consider the infinite product of restricted 2-colored partition function with product of three factors $P_{2,3}(m)$ is defined by

$$F(q) = \sum_{m\geq 0} P_{2,3}(m) = \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^2(1-q^{4m-2})^2(1-q^{4m})^2} \quad (18)$$

which integrates three distinct arithmetic sequences, each appearing with multiplicity two, into a single generating function. The first factor represents partitions with two colors over all positive integers, the second factor corresponds to two-colored partitions over 2, 6, 10, …and the third factor covers multiples of four.

To the best of our knowledge, the infinite q-product $F(q)$ and the combinatorial and analytic results presented here are original and have not appeared in previous literature.



The surface plot highlights the dominant singular behavior of $F(q)$ near $q = 1$, while confirming bounded and analytic behavior within the unit disk. These features are consistent with the derived asymptotic growth and validate the analytic framework used in this study.

Theorem 4.1

Let $\{P_{2,3}(m)\}$ be the sequence of 2-colored restricted partition function with product of three factors. Then the general term of the sequence $\{P_{2,3}(m)\}$ is

$$P_{2,3}(m) = \sum_{\substack{a_n, b_n, c_n \geq 0 \\ \sum n a_n + \sum (4n-2) b_n + \sum 4n c_n = m}} \prod_{n \geq 1} (a_n + 1)(b_n + 1)(c_n + 1), m \geq 0. \quad (19)$$

The triple convolution recursive formula of this sequence is

$$P_{2,3}(m) = \sum_{k=1}^{m} \sum_{i=0}^{k} p_2(i)\, b(k-i) c(m-k), \quad P_{0,3} = 1 \quad (20)$$

The equation (20) provides a combinatorial interpretation and $P_{2,3}(m)$ counts generalized 2-colored partitions of m using three distinct arithmetic progressions with multiplicity two.

This allows efficient computation of coefficients and reflects the interplay of the three sequences.

4.1. Analytical Properties and Asymptotic Behavior

The infinite product $F(q)$ admits several analytic interpretations, providing insights into the growth of its coefficients $\{P_{2,3}(m)\}$.

4.1.1. Radius of Convergence and Singularities

Since each factor $(1-q^s)^{-2}$ of infinite product $F(q)$ is analytic for $|q| < 1$. The dominant singularity at $|q| = 1$ governs asymptotic growth. Singularities at roots of unity are subdominant and do not affect leading-order asymptotic.

4.1.2. Relation to Eta Functions and Modular Forms.

Each factor $(1-q^s)^{-2}$ we can be expressed as Dedekind eta function $\varphi(\tau)$, that is $\frac{1}{(1-q^s)^2} \sim \frac{1}{\varphi(s\tau)^2}$ up to multiplicative powers of q.

Hence

$$F(q) \sim \frac{1}{\varphi(\tau)^2 \varphi(2\tau)^2 \varphi(4\tau)^2}$$

4.1.3. Asymptotic Growth of Coefficients

By Tauberian and saddle-point methods, the partition $P_{2,3}(m)$ can be derived as

$P_{2,3}(m) \sim A\, m^{-\alpha} \exp(\pi\sqrt{\beta m}),\ m \to \infty$, with constants $A, \alpha, \beta > 0$

5. Conclusion

Analytic and computational results were established for a weighted partition generating function defined by a structured Euler product. Explicit coefficient formulas, Euler-type recurrence relations, and asymptotic growth estimates were derived. Truncated-product computations exhibited stable behaviour inside the unit disk and rapid growth near the unit-circle boundary, consistent with the predicted dominant singular behaviour. Future research will consider more general Euler products, congruence properties modulo primes and prime powers, and sharper truncation-error bounds for accurate evaluation near the unit circle..

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