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A Study of Fundamental Parameters in Linear Error-Correcting Codes Associated with $A(n, k)$

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Abstract

In this work, q-ary linear codes are constructed from row space of $|V| \times |E|$ *generator matrix of* $A(n,k)$, (n,k)-arrangement graph where n and k are natural numbers satisfying $n > k+1$. The combinatorial structure of arrangement graphs makes them a suitable source for generating well-organized linear codes through their incidence matrices. We investigate the algebraic properties of these codes and determine their key parameters, namely the minimum distance, dimension, and length. These parameters are shown to be $\left[\frac{k}{2}\prod_{i=0}^{k}(n-i),\ \prod_{i=0}^{k-1}(n-i),\ k(n-k)\right]_q$. Further, we also discuss the linear code obtained from the Möbius–Kantor graph, which yields a linear code with parameters [24, 15, 3]. All codes obtained in this manner are linear over the finite field F_q. Each transitive subgroup of the group of automorphism provides a permutation decoding set for the associated codes, ensuring that full error correction can be achieved through permutation decoding.

1 Introduction

Permutation decoding, introduced by MacWilliams in [14], provides a method for decoding received vectors by exploiting suitable subsets of the group of automorphism of a code. In that work, a systematic approach was developed to identify PD-sets for cyclic codes. A subset $S \subseteq F_q^n$ is said to be cyclic if, whenever $(a_0, a_1, \dots, a_{n-1}) \in S$, its cyclic shift $(a_{n-1}, a_0, a_1, \dots, a_{n-2})$ also belongs to S. Later, Chabanne [8] obtained PD-sets for abelian codes. Although permutation decoding was initially studied for classical families such as Hamming and Golay codes, it has since been extended to codes with large automorphism groups. In general, determining PD-sets remains a challenging problem, and Gordon established a lower bound on their minimum size.

In [7] Goodman and Green constructed both hard and soft-decision permutation decoding algorithms using an Intel 8080A microprocessor. Gordon [8] and Wolfman [20] have constructed small PD-sets for binary Golay codes. Similarly, [12] examined PD-sets for generalized Reed-Muller codes.

Finding a suitable information set is an essential step because the development of permutation decoding sets significantly depends on the selection of an information set.

Codes from incidence and adjacency matrices of graphs and combinatorial designs often provide natural candidates for both information sets and PD-sets. A comprehensive survey of permutation decoding can be found in [19].

Key, Fish, and Mwambene created q-ary linear codes from the generator matrix $G_n(m)$ in [5, 11] of the Hamming graph $H(n,m)$ with parameters

$$C_q\big(G_n(m)\big) = \{\left[\frac{1}{2}m^n(m-1)n,\ m^n,\ (m-1)n\right]_q, \quad if\ q\ is\ odd, \quad \left[\frac{1}{2}m^n(m-1)n,\ m^n-1,\ (m-1)n\right]_2, \quad if\ q = 2,$$

and also described codes obtained from the incidence matrix of $H^k(n,2)$ for $1 \le k < n$, given by

$$C_q\big(G_n^k\big) = \{\left[2^{n-1}\binom{n}{k},\ 2^n - 1,\ \binom{n}{k}\right]_q, \quad if\ k\ is\ odd, \left[2^{n-1}\binom{n}{k},\ 2^n - 1,\ \binom{n}{k}\right]_q, \quad if\ k\ is\ even\ and\ q = 2, \left[2^{n-1}\binom{n}{k},\ 2^n,\ \binom{n}{k}\right]_q, \quad if\ k\ is\ even\ and\ q\ is\ odd.$$

In [5], the following useful result was established.s

Proposition 1. Let $\Gamma = (V,E)$ be a k-regular graph whose automorphism group A acts transitively on its edges, and let M denote its incidence matrix. For a prime q, condiset the linear code $C = C_q(M)$ with parameters $[|E|, |V| - \epsilon, k]_q$, where $\epsilon \in \{0,1,\dots,|V|-1\}$. Then any transitive subgroup of A give rise to a PD-set that enable full error correction of the code C.

In [3], the same authors derived the binary code $[3^n, \frac{1}{2}(3^n - (-1)^n), 2n]_2$ from the adjacency matrix of the neighbourhood design $D(n,3)$ associated with $H(n,3)$, and also studied PD-sets for partial permutation decoding.

Furthermore, using the incidence matrix of the entire graph K_n, Key, Moori, and Rodrigues created q-ary linear codes with parameters in [13],

$$[\binom{n}{2}, n, n-1]_q.$$

They found $I_n = \{[1,n],\dots,[n-1,1],1,2]\}$ an information set and $S = \{(n,i)(1,j) \mid 1 \le i \le n,\ 1 \le j \le n-1\} \subseteq S_n$ its corresponding PD-set. Additionally, for $n \ge 5$, they derived codes with the same parameters from $G_n(1)$ of $A(n,1)$.

Note: Notation $Z_{a,b} = \{i \in N \mid a \le i \le b\}$.

The construction of special classes of codes and their applications has been explored in recent works such as [15, 16], where different families of linear codes are developed and their properties are studied.

In [1], Dankelmann, Key, and Rodrigues examined the codes obtained from the incidence matrices of connected graphs with n vertices in which each vertex has degree k. They proved that these codes have a dimension of either n or $n-1$ and a minimum weight of k. Furthermore, the minimum-weight codewords are scalar multiples of the rows in the incidence matrix. The following result was reached after they discovered a gap in the weight distribution between k and $2k-2$.

Proposition 2. Consider a connected graph $\Gamma = (V,E)$ with incidence matrix I. Let $C_q(I)$ denote the code generated by the rows of I over F_q. Then

$$dim\big(C_2(I)\big) = |V| - 1,$$

and for odd prime q,

$$dim\left(C_q(I)\right) = \{|V|, \quad if\ \Gamma\ is\ not\ bipartite, |V|-1, \quad if\ \Gamma\ is\ bipartite.$$

In [18], codes with parameters

$$\left[\prod_{i=0}^{2} (n-i), \prod_{i=0}^{1} (n-i), 2(n-2)\right]_q$$

were obtained from $G_n(2)$ of $A(n,2)$ for $n \geq 4$.

Motivated by these developments and the combinatorial richness of arrangement graphs, we investigate the generator matrices of $A(n,k)$ for $n > k+1$. From these matrices, we construct linear codes and determine their fundamental parameters. In particular, we examine the codes for $A(n,2)\ for\ n > 3$, $A(n,3)\ for\ n > 4$ and $A(n,4)\ for\ n > 5$. In addition, we examine the code arising from the Möbius–Kantor graph which has parameters [24, 15, 3]. This work highlights the interplay between graph-theoretic structures and coding theory via incidence matrix constructions.

2 Preliminaries

Let F_q be a finite field consisting of q elements. Consequently, F_q^n is a vector space over F_q with n dimensions. The Hamming weight $\delta(x)$ for a vector $x = (x_1, x_2, \dots, x_n) \in F_q^n$ is defined as follows

$$\delta(x) = \sum_{j=1}^{n} \delta(x_j), \quad where \quad \delta(x_j) = \{1, \quad if\ x_j \neq 0, 0, \quad if\ x_j = 0.$$

For vectors $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ in F_q^n, the Hamming distance between them is given by

$$d(u,v) = \sum_{j=1}^{n} \delta(u_j - v_j).$$

A q-ary code of length n is any non-empty subset of F_q^n, whose elements are called codewords. It is called linear if this set forms a subspace of F_q^n.

The minimum weight $\delta(C)$ is the least Hamming weight among all non-zero codewords in C. For linear codes, $d(C) = \delta(C)$.

Proposition 3. [17] Let H be a parity-check matrix of a linear code C. Then C has minimum distance d if and only if:

- Any set of $d-1$ columns of H are linearly independent,
- There exists a set of d columns of H that is linearly dependent.

Throughout this paper, we consider undirected, simple, and regular graphs, and all codes discussed are linear. Let $\Gamma = (V,E)$ be a graph. Its incidence matrix $I = [a_{ij}]$ is a $|V| \times |E|$ matrix in which rows correspond to vertices and columns correspond to edges. The entry a_{ij} indicates the i^{th} vertex is incident with the j^{th} edge.

$$a_{ij} = \{1, \quad if\ \ e_j\ is\ incident\ with\ the\ v$$

$[n,k,d]_q$ represents a q-ary linear code with dimension k, minimum distance d, and length n. The rows of a generator matrix G of a linear code C forms a basis for C. The notation $C_q(G)$ denotes the code generated by G over F_q, and its dimension equals the rank of G.

We now introduce some terminology related to configurations.

Definition 4. A (v_r, b_k)-configuration is a configuration such that:

1. $|P| = v$ with $v \geq 4$,
2. $|L| = b$ with $b \geq 4$,
3. each line is incident with exactly k points ($k \geq 2$),
4. each point is incident with exactly r lines ($r \geq 2$).

If $v = b$ and $r = k$, it is denoted by (v_r).

2.1 Arrangement Graph

An arrangement graph is defined as follows. All k-permutations of n symbols form its vertex set, and two vertices are neighboring if they differ in exactly one position. Thus, the edge set is $E = \{[x, y] \mid d(x, y) = 1\}$.

The graph $A(n, k)$ is $k(n-k)$-regular. It contains $\frac{n!}{(n-k)!}$ vertices, and consequently its number of edges is given by $\frac{k}{2}\frac{n!(n-k)}{(n-k)!}$.

Proposition 5. [2] The graph $A(n, k)$ has connectivity equal to $k(n-k)$.

The generator matrix $G_n(k)$ for $A(n, k)$ is of size$\frac{n!}{(n-k)!} \times \frac{k}{2}\frac{n!(n-k)}{(n-k)!}$

To describe its structure, we first arrange the rows according to the vertices. These rows are grouped into n blocks X_i, where each block consists of all k-tuples whose last coordinate is i.

The columns, corresponding to edges, are ordered systematically. We begin with edges joining vertices within the same block, listing those in X_1, followed by X_2, and so on up to X_n. After that, edges between different blocks are included in a sequential manner, starting with pairs such as (X_1, X_2), ..., (X_1, X_{n-1}), (X_2, X_3), ..., (X_2, X_n), and continuing until all inter-block edges are accounted for. As a result, the graph $G_n(k)$ admits a decomposition into $2n-1$ column blocks and n row blocks.

2.2 Construction of the generator matrix for $A(n, 1)$

For $A(n, 1)$, there are n vertices and $\binom{n}{2}$ edges. It is shown in [13] that $G_2(1) = (1)$. In general, the generator matrix satisfies the recursive relation

$$\mathcal{G}_n(1) = \left(\begin{array}{c|c} 1 \cdots 1 & 0 \cdots 0 \\ \hline \mathcal{I}_{n-1} & \mathcal{G}_{n-1}(1) \end{array}\right)$$

where the identity matrix of order $n-1$ is represented by I_{n-1}. Therefore, $G_n(1)$ is a $n \times \binom{n}{2}$ matrix.

For this graph, every pair of unique vertices is adjacent to the vertex set. Therefore, $A(n, 1)$ is a complete graph with n vertices.

Example 6. Consider $A(5,1)$ and its generator matrix $G_5(1)$.

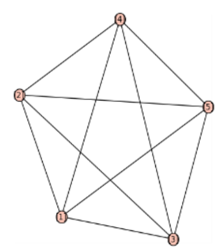

Figure 1: (5,1)-arrangement graph

$$\mathcal{G}_5(1) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

3 Construction and Structure of the generator Matrix of $A(n, k)$

Every integer n with $n > k + 1$, we examine error-correcting codes derived from the generator matrix of $A(n, k)$. |V| and |E| of $A(n, k)$ is $\frac{n!}{(n-k)!} = \prod_{i=0}^{k-1} (n-i)$ and $\frac{k}{2}\prod_{i=0}^{k} (n-i)$ respectively. Consequently, the size of the corresponding generator matrix $G_n(k)$ is

$$\prod_{i=0}^{k-1} (n-i) \times \frac{1}{2}\prod_{i=0}^{k} (n-i)k(n-k).$$

The matrix $G_n(k)$ admits a natural block structure that can be described recursively in terms of $G_{n-1}(k-1)$. In particular, it can be expressed in the following block for

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c}
\mathcal{G}_{n-1}(k-1) & 0 & 0 & \cdots & 0 & 0 & 0 & \mathcal{I}_1 & 0' & 0'' & \cdots & 0^{(n-3)} & 0^{(n-2)} \\ \hline
0 & \mathcal{G}_{n-1}(k-1) & 0 & \cdots & 0 & 0 & 0 & \mathcal{B}_1' & \mathcal{I}_2 & 0'' & \cdots & 0^{(n-3)} & 0^{(n-2)} \\ \hline
0 & 0 & \mathcal{G}_{n-1}(k-1) & \cdots & 0 & 0 & 0 & \mathcal{B}_1'' & \mathcal{B}_2' & \mathcal{I}_3 & \cdots & 0^{(n-3)} & 0^{(n-2)} \\ \hline
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \hline
0 & 0 & 0 & 0 & \mathcal{G}_{n-1}(k-1) & 0 & 0 & \mathcal{B}_1^{(n-3)} & \mathcal{B}_2^{(n-4)} & \mathcal{B}_3^{(n-5)} & \cdots & \mathcal{I}_{n-2} & 0^{(n-2)} \\ \hline
0 & 0 & 0 & 0 & 0 & \mathcal{G}_{n-1}(k-1) & 0 & \mathcal{B}_1^{(n-2)} & \mathcal{B}_2^{(n-3)} & \mathcal{B}_3^{(n-4)} & \cdots & \mathcal{B}_{n-1}' & \mathcal{I}_{n-1} \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & \mathcal{G}_{n-1}(k-1) & \mathcal{B}_1^{(n-1)} & \mathcal{B}_2^{(n-2)} & \mathcal{B}_3^{(n-3)} & \cdots & \mathcal{B}_{n-2}'' & \mathcal{B}_{n-1}'
\end{array}\right)$$

Here, O denotes a zero matrix of order $\prod_{i=0}^{k-2} \quad (n-i-1) \times \frac{k-1}{2}\prod_{i=0}^{k-1} \quad (n-i-1)$, $\forall\ j \in Z_{2,n-1}$ the matrix $O^{(j-1)}$ represents a zero matrix of size $\prod_{i=0}^{k-2} \quad (n-i-1) \times (n-j)\prod_{i=2}^{k} \quad (n-i)$.

The matrices I_j, defined for $j \in Z_{2,n-1}$, have the same dimensions as $O^{(j-1)}$ and follow a structured pattern. In particular, each row in I_1 has exactly $(n-k)$ diagonally ordered entries equal to 1, remaining all zero.

$$\mathcal{I}_1 = \begin{pmatrix}
\underbrace{11\ldots1}_{(n-k)} & 0 & 0 & \cdots & 0 \\
0 & \ddots & 0 & & \vdots \\
0 & 0 & \underbrace{11\ldots1}_{(n-k)} & 0 & 0 \\
\vdots & & 0 & \ddots & 0 \\
0 & \cdots & 0 & 0 & \underbrace{11\ldots1}_{(n-k)}
\end{pmatrix}.$$

For $j \in Z_{2,n-1}$, the matrices I_j are defined similarly, although the placement of the non-zero entries depends on the matrices $B_m^{(l)}$, where $m, l \in j \in Z_{1,j-2}$ and $m + l = j$. In this case, at most $(n-k)$ items in each row of I_j are 1, whereas all other are zero. The matrices $B_j^{(m)}$, defined for $j \in Z_{1,n-2}$, have the same dimensions as I_j and satisfy the property that each row contains at most one non-zero entry, which is equal to 1.

Thus, the matrix $G_n(k)$ as being composed of $2n-1$ column blocks and n row blocks. The matrices I_j, $B_j^{(m)}$, and $O^{(j-1)}$ are used to build the remaining $n-1$ column blocks, while copies of $G_{n-1}(k-1)$ are present along the diagonal of the first n column blocks. Using $C_1, C_2, \ldots, C_{2n-1}$ to represent the column blocks, we get the following outcome.

Theorem 7. For $n > k+1$, $A(n,k)$ is a regular graph with $\prod_{i=0}^{k-1} \quad (n-i)$ vertices and $\frac{k}{2}\prod_{i=0}^{k} \quad (n-i)$ edges.

Since $A(n,k)$ are symmetric, proposition 1 yields the following consequence.

Theorem 8. The group of automorphism of $A(n,k)$, when $n > k+1$, is represented by S. Consequently, each transitive subgroup of S functions as a permutation decoding set, guaranteeing complete error-correcting capacity for the code $C_q\big(G_n(k)\big)$.

4 Construction of the generator matrix for $A(n,2)$

For all integers $n > 3$, we analyze the codes resulting from the generator matrix of $A(n,2)$. The graph $A(n,2)$ consists of $\prod_{i=0}^{1} \quad (n-i)$ vertices and $\prod_{i=0}^{2} \quad (n-i)$ edges. Accordingly, the associated generator matrix $G_n(2)$ has $\prod_{i=0}^{1} \quad (n-i)$ rows and $\prod_{i=0}^{2} \quad (n-i)$ columns. Hence, this matrix admits the following block structure.

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c}
\mathcal{G}_{n-1}(1) & 0 & 0 & \dots & 0 & 0 & 0 & \mathcal{I}_1 & 0' & 0'' & \dots & 0^{(n-3)} & 0^{(n-2)} \\ \hline
0 & \mathcal{G}_{n-1}(1) & 0 & \dots & 0 & 0 & 0 & \mathcal{B}'_1 & \mathcal{I}_2 & 0'' & \dots & 0^{(n-3)} & 0^{(n-2)} \\ \hline
0 & 0 & \mathcal{G}_{n-1}(1) & \dots & 0 & 0 & 0 & \mathcal{B}''_1 & \mathcal{B}'_2 & \mathcal{I}_3 & \dots & 0^{(n-3)} & 0^{(n-2)} \\ \hline
\vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \hline
0 & 0 & 0 & 0 & \mathcal{G}_{n-1}(1) & 0 & 0 & \mathcal{B}_1^{(n-3)} & \mathcal{B}_2^{(n-4)} & \mathcal{B}_3^{(n-5)} & \dots & \mathcal{I}_{n-2} & 0^{(n-2)} \\ \hline
0 & 0 & 0 & 0 & 0 & \mathcal{G}_{n-1}(1) & 0 & \mathcal{B}_1^{(n-2)} & \mathcal{B}_2^{(n-3)} & \mathcal{B}_3^{(n-4)} & \dots & \mathcal{B}'_{n-1} & \mathcal{I}_{n-1} \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & \mathcal{G}_{n-1}(1) & \mathcal{B}_1^{(n-1)} & \mathcal{B}_2^{(n-2)} & \mathcal{B}_3^{(n-3)} & \dots & \mathcal{B}''_{n-2} & \mathcal{B}'_{n-1}
\end{array}\right)$$

In this representation, O denotes a zero matrix of size $(n-1) \times \frac{1}{2}(n-1)(n-2)$. For each j with $j \in Z_{2,n-1}$, the matrix $O^{(j-1)}$ is a zero matrix of order $(n-1) \times (n-j)(n-2)$.

The matrices I_j, defined for $j \in Z_{1,n-1}$, are of order $(n-1) \times (n-j)(n-2)$ and follow a specific structural pattern. In particular, the matrix I_1 has $(n-1)$ rows, each containing exactly $(n-2)$ entries equal to 1. These non-zero entries are arranged diagonally, while all remaining entries are zero. For I_j with $j \in Z_{2,n-1}$, the placement and number of ones in each row depend on the contributions from the matrices $B_m^{(l)}$, where $m, l \in Z_{1,j-2}$ with $m + l = j$.

Consequently, each row of I_j contains at most $(n-2)$ ones, with all other entries equal to zero. The matrices $B_j^{(m)}$, defined for $j \in Z_{2,n-1}$, are of order $\prod_{i=0}^{1} \quad (n-i-1) \times (n-j) \prod_{i=2}^{3} \quad (n-i)$. Each of these matrices is sparse, containing at most one non-zero entry in any row.

Therefore, the generator matrix $G_n(2)$ can be decomposed as n rows and the first n column blocks consist of copies of $G_{n-1}(1)$ placed along the diagonal, while the remaining $n-1$ column blocks are formed using the matrices I_j, $B_j^{(m)}$, and $O^{(j-1)}$. Denoting these column blocks by $C_1, C_2, \dots, C_{2n-1}$, we obtain the desired block decomposition of $G_n(2)$.

Example 9. Consider the graph $A(4,2)$ and its generator matrix. The cardinality of vertices in this graph is 12, and the cardinality of edges is 24.

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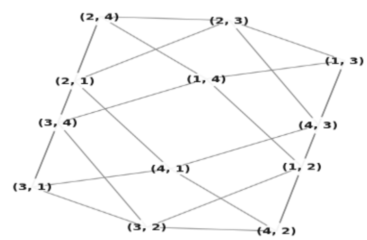

Figure 2: $(4,2)$*-arrangement graph.*

The corresponding generator matrix $G_4(2)$ is an 12×24 matrix with a block structure that reflects the arrangement of vertices and edges. It can be expressed in the following form

$$\left(\begin{array}{ccc|ccc|ccc|ccc|cccccc|cccc|cc}
1&1&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
\hline
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&1&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0\\
\hline
0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0\\
\hline
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&1&0&0&1&0\\
0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&1&0&0&0&0&1&0&0&0
\end{array}\right)$$

Thus, the following results are obtained, which describe the structural properties of $A(n,2)$ and are useful in finding the parameters of the code from its generator matrix.

Lemma 10. The generator matrix of $A(n-1,1)$ is contained in each row block of $G_n(2)$ in $A(n,2)$. Hence, for any codeword $c \in C_q(G_n(2))$, the component corresponding to a row block belongs to the code $C_q(G_{n-1}(1))$.

Theorem 11. For every integer $n > 3$, $A(n,2)$ is a regular graph with $\prod_{i=0}^{1} (n-i)$ vertices and $\prod_{i=0}^{2} (n-i)$ edges.

4.1 Fundamental Parameters of Error-Correcting Codes from $A(n,2)$

For $n > 3$, we study the linear codes $C_q(G_n(2))$ derived from $G_n(2)$ of $A(n,2)$ and discuss their fundamental parameters.

Theorem 12. Let $G_n(2)$ be $A(n,2)$'s generator matrix. The linear code $C_q(G_n(2))$ for every integer $n > 3$ and any odd prime q produced by $G_n(2)$ is a q-ary linear code with parameters

$$\left[\prod_{i=0}^{2} (n-i),\ \prod_{i=0}^{1} (n-i),\ 2(n-2)\right]_q.$$

Proof. We examine the code length. The $A(n,2)$ has $\prod_{i=0}^{1} (n-i)$ vertices and $\prod_{i=0}^{2} (n-i)$ edges. Hence, its generator matrix $G_n(2)$ has size $\prod_{i=0}^{1} (n-i) \times \prod_{i=0}^{2} (n-i)$.

Since $G_n(2)$ serves as a generator matrix for $C_q(G_n(2))$, the code length = $|E|$ of $G_n(2) = \prod_{i=0}^{2} (n-i)$.

Then determine the dimension of the code for $A(n,2)$, since it is connected, for $n > 3$ it contains an odd cycle; for example, $(1,2) \to (1,3) \to (1,4) \to (1,2)$. Hence, the graph is non-bipartite. Since q is an odd prime, the field F_q has characteristic different from 2.

Therefore, the generator matrix of a connected non-bipartite graph has full row rank over F_q.
It follows that

$$\begin{aligned} dim\left(C_q\left(G_n(2)\right)\right) &= |V| \text{ in } A(n,2) \\ &= \prod_{i=0}^{1} (n-i). \end{aligned}$$

Next, use induction on n to find the smallest weight of $C_q\left(G_n(2)\right)$. The generator matrix corresponding to the $A(5,2)$ is,

$$\mathcal{G}_5(2) = \left(\begin{array}{c|c|c|c|c|c|c|c|c} \mathcal{G}_4(1) & 0 & 0 & 0 & 0 & \mathcal{I}_1 & 0' & 0'' & 0''' \\ \hline 0 & \mathcal{G}_4(1) & 0 & 0 & 0 & \mathcal{B}_1' & \mathcal{I}_2 & 0'' & 0''' \\ \hline 0 & 0 & \mathcal{G}_4(1) & 0 & 0 & \mathcal{B}_1'' & \mathcal{B}_2' & \mathcal{I}_3 & 0''' \\ \hline 0 & 0 & 0 & \mathcal{G}_4(1) & 0 & \mathcal{B}_1''' & \mathcal{B}_2'' & \mathcal{B}_3' & \mathcal{I}_4 \\ \hline 0 & 0 & 0 & 0 & \mathcal{G}_4(1) & \mathcal{B}_1^{iv} & \mathcal{B}_2''' & \mathcal{B}_3'' & \mathcal{B}_4' \end{array}\right)$$

$$\mathcal{G}_5(2) = \left(\begin{array}{cccccc|cccccc|cccccc|cccccc|cccccc|cccccccccccc|ccccccccc|cccccc|ccc}
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0
\end{array}\right)$$

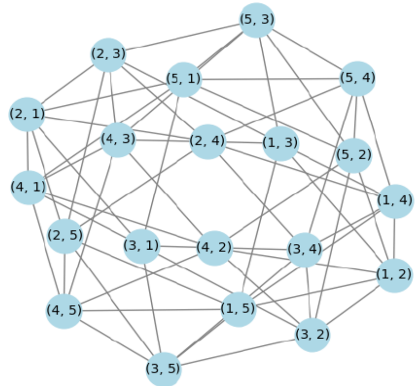

Figure 4: (5,2)-arrangement graph.

Base case: $n = 5$. We show that $\omega\left(C_q\left(G_5(2)\right)\right) = 6$. Consider the generator matrix $G_5(2)$ written in block form. Partition its rows into blocks X_i, $i \in Z_{1,5}$, and columns into blocks C_j, $j \in Z_{1,9}$. Let $c \in C_q\left(G_5(2)\right)$, and write $c = (c_1, c_2, \dots, c_9)$, c_j corresponds to the block C_j. Then

$$\omega(c) = \sum_{j=1}^{9} \omega(c_j).$$

Let c be a linear combination of rows from a single block X_1. If c is a row of $G_5(2)$, then $\omega(c) = 6$. Otherwise, since $c_1 \in C_q\left(G_4(1)\right)$, we have $\omega(c_1) \geq \omega\left(C_q\left(G_4(1)\right)\right) = 3$. The component c_6 arises from the identity block I_1. Each row of I_1 has exactly 3 ones, and these supports are disjoint. Hence, if r rows are selected, then $\omega(c_6) = 3r \geq 3$. Thus, $\omega(c) = \omega(c_1) + \omega(c_6) \geq 3 + 3 = 6$. A similar argument applies to any single block X_i, $i \in Z_{2,5}$, where the contributions from the corresponding column blocks ensure that $\omega(c) \geq 6$.

Let us now assume that c lies in the span of rows from two different blocks, X_1 and X_2. Then $c_1, c_2 \in C_q\left(G_4(1)\right)$, so $\omega(c_1), \omega(c_2) \geq 3$. Since these correspond to disjoint column blocks, $\omega(c) \geq \omega(c_1) + \omega(c_2) \geq 6$.

Thus, each non-zero codeword has a minimum weight of 6, and equality is attained by rows of $G_5(2)$. Therefore, $\omega\left(C_q\left(G_5(2)\right)\right) = 6$.

Inductive step: Assume that $\omega\left(C_q\left(G_{n-1}(2)\right)\right) = 2(n-3)$.

We now demonstrate that $\omega\left(C_q\left(G_n(2)\right)\right) = 2(n-2)$. Let $c \in C_q\left(G_n(2)\right)$. c is a generator vector of the graph if it is a row of $G_n(2)$. Since $A(n,2)$ is regular of degree $2(n-2)$, each row contains exactly $2(n-2)$ non-zero entries. Hence, there exists a codeword with weight $2(n-2)$.

Now let c be any non-zero codeword. In case, c is a linear sum of rows from a single block X_i. Then $c_i \in C_q\left(G_{n-1}(1)\right)$. It is known that $\omega\left(C_q\left(G_{n-1}(1)\right)\right) = (n-1) - 1$. Hence, $\omega(c_i) \geq n - 2$. From the structure of the matrices I_j and $B_m^{(l)}$, each selected row contributes at least one non-zero entry to distinct column blocks outside C_i, and these contributions do not cancel. It follows that

$$\omega(c_{n+1}) + \cdots + \omega(c_{2n-1}) \geq n - 2.$$

Hence,

$$\omega(c) \geq 2(n-2).$$

If c is a linear sum of rows from two or more distinct row blocks, then at least two of the components c_i are not zero. Each such component lies in $C_q\big(G_{n-1}(1)\big)$ and satisfies $\omega(c_i) \geq n-2$. Since these correspond to disjoint column blocks, their contributions add, and hence $\omega(c) \geq 2(n-2)$. Consequently, this bound is reached and the weight of non-zero codeword in $C_q\big(G_n(2)\big)$ is at least $2(n-2)$. Hence,

$$\omega\left(C_q\big(G_n(2)\big)\right) = 2(n-2).$$

Thus, the code $C_q\big(G_n(2)\big)$ has parameters

$$\left[\prod_{i=0}^{2} (n-i),\ \prod_{i=0}^{1} (n-i),\ 2(n-2)\right]_q.$$

□

Using Proposition 1, the following theorem explains how the automorphism group of $A(n,2)$ can be used to obtain a PD-set for permutation decoding.

Theorem 13. For $n > 3$, let S represent the group of automorphism of $A(n,2)$. For the code $C_q\big(G_n(2)\big)$, each transitive subgroup of S forms a permutation decoding set that ensures complete error correction.

5 Construction of the generator matrix for $A(n,3)$

We determine the linear codes for all integers $n > 4$ derived from the generator matrix of $A(n,3)$. The graph has $\prod_{i=0}^{2} (n-i)$ vertices, and $\frac{3}{2}\prod_{i=0}^{3} (n-i)$ edges. Consequently, the associated generator matrix $G_n(3)$ consists of $\prod_{i=0}^{2} (n-i)$ rows and $\frac{3}{2}\prod_{i=0}^{3} (n-i)$ columns. With this, the matrix $G_n(3)$ admits the following structured block representation.

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c}
\mathcal{G}_{n-1}(2) & 0 & 0 & \dots & 0 & 0 & 0 & \mathcal{I}_1 & 0' & 0'' & \dots & 0^{(n-3)} & 0^{(n-2)} \\ \hline
0 & \mathcal{G}_{n-1}(2) & 0 & \dots & 0 & 0 & 0 & \mathcal{B}_1' & \mathcal{I}_2 & 0'' & \dots & 0^{(n-3)} & 0^{(n-2)} \\ \hline
0 & 0 & \mathcal{G}_{n-1}(2) & \dots & 0 & 0 & 0 & \mathcal{B}_1'' & \mathcal{B}_2' & \mathcal{I}_3 & \dots & 0^{(n-3)} & 0^{(n-2)} \\ \hline
\vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \hline
0 & 0 & 0 & 0 & \mathcal{G}_{n-1}(2) & 0 & 0 & \mathcal{B}_1^{(n-3)} & \mathcal{B}_2^{(n-4)} & \mathcal{B}_3^{(n-5)} & \dots & \mathcal{I}_{n-2} & 0^{(n-2)} \\ \hline
0 & 0 & 0 & 0 & 0 & \mathcal{G}_{n-1}(2) & 0 & \mathcal{B}_1^{(n-2)} & \mathcal{B}_2^{(n-3)} & \mathcal{B}_3^{(n-4)} & \dots & \mathcal{B}_{n-1}' & \mathcal{I}_{n-1} \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & \mathcal{G}_{n-1}(2) & \mathcal{B}_1^{(n-1)} & \mathcal{B}_2^{(n-2)} & \mathcal{B}_3^{(n-3)} & \dots & \mathcal{B}_{n-2}'' & \mathcal{B}_{n-1}'
\end{array}\right)$$

Here, zero matrix O is of order $\prod_{i=0}^{1} (n-i-1) \times \prod_{i=0}^{2} (n-i-1)$. Similarly, for $j \in Z_{2,n-1}$, the matrix $O^{(j-1)}$ represents a zero matrix of size $\prod_{i=0}^{1} (n-i-1) \times (n-j)\prod_{i=2}^{3} (n-i)$. For each $j \in Z_{2,n-1}$, the matrix I_j has the same order $\prod_{i=0}^{1} (n-i-1) \times (n-j)\prod_{i=2}^{3} (n-i)$. The matrix I_1 is given by

$$\mathcal{I}_1 = \begin{pmatrix} \underbrace{11\ldots1}_{(n-3)} & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & & \vdots \\ 0 & 0 & \underbrace{11\ldots1}_{(n-3)} & 0 & 0 \\ \vdots & & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \underbrace{11\ldots1}_{(n-3)} \end{pmatrix}$$

In particular, each row of I_1 contains exactly $(n-3)$ entries equal to 1, arranged diagonally across the $\prod_{i=0}^{1} \quad (n-i-1)$ rows. For the matrices I_j with $j \in Z_{2,n-1}$, the entries are constructed using auxiliary matrices $B_m^{(l)}$, $m, l \in \ Z_{1,j-2}$ with $m+l=j$. In these matrices, each row contains at most $n-3$ ones, and all remaining entries are zero.

Every matrix $B_m^{(l)}$, satisfying $m, l \in \ Z_{1,j-2}$ with $m+l=j$, has order

$$\prod_{i=0}^{1} \quad (n-i-1) \times (n-j) \prod_{i=2}^{3} \quad (n-i),$$

and every row of $B_j^{(m)}$ has at most one entry equal to 1. The generator matrix $G_n(3)$ admits a block decomposition in which the initial set of column blocks consists of diagonal copies of $G_{n-1}(2)$. The remaining column blocks are constructed using the matrices I_j, $B_m^{(l)}$, and $O^{(j-1)}$. For convenience, we label the column blocks of $G_n(3)$ as $C_1, C_2, \ldots, C_{2n-1}$.

The following lemmas define the structural features of $A(n,3)$ and are helpful in identifying the parameters of the code produced by $G_n(3)$.

Lemma 14. Each row block of $G_n(3)$ in $A(n,3)$ has a submatrix that is the generator matrix of $A(n-1,2)$. Hence, for any codeword $c \in C_q\big(G_n(3)\big)$, the component corresponding to a row block belongs to the code $C_q\big(G_{n-1}(2)\big)$.

Theorem 15. For $n > 4$, $A(n,3)$ forms a regular graph with $\prod_{i=0}^{2} \quad (n-i)$ vertices and $\frac{3}{2}\prod_{i=0}^{3} \quad (n-i)$ edges.

5.1 Fundamental Parameters of Error-Correcting Codes from $A(n,3)$

Here, we study the linear codes $C_q\big(G_n(3)\big)$ derived from $G_n(3)$ of $A(n,3)$ for all integers $n > 4$, with a focus on determining their key parameters.

Theorem 16. Let $G_n(3)$ be $A(n,3)'$s generator matrix. The associated code $C_q\big(G_n(3)\big)$ is a q-ary linear code with parameters for any odd prime q and any integer $n > 4$,

$$\left[\frac{3}{2}\prod_{i=0}^{3} \quad (n-i),\ \prod_{i=0}^{2} \quad (n-i),\ 3(n-3)\right]_q.$$

Proof. We begin by determining the code length. As $G_n(3)$ represents the generator matrix of $A(n,3)$, the graph has $\prod_{i=0}^{2} \quad (n-i)$ vertices and $\frac{3}{2}\prod_{i=0}^{3} \quad (n-i)$ edges. Consequently, the generator matrix $G_n(3)$ has size

$$\prod_{i=0}^{2} \quad (n-i) \times \frac{3}{2}\prod_{i=0}^{3} \quad (n-i).$$

The code length is number of columns of $A(n,3) = \frac{3}{2}\prod_{i=0}^{3} \quad (n-i)$.

By proposition 5, $A(n,3)$ is connected. For $n > 4$, choose distinct elements $a, b, c, d, e \in \{1,2,\dots,n\}$. Consider the vertices $p_1 = (a,b,c)$, $p_2 = (a,b,d)$, $p_3 = (a,b,e)$. These vertices are pairwise adjacent, and hence form the cycle $p_1 \to p_2 \to p_3 \to p_1$. Thus, $A(n,3)$ contains an odd cycle for all $n > 4$.

Since q is an odd prime, the field F_q has characteristic different from 2. Therefore, by Proposition 2, the generator matrix $G_n(3)$ has full row rank, as the graph is connected and contains an odd cycle. Hence,

$$dim\left(C_q\big(G_n(3)\big)\right) = |V| \; in \; A(n,3) = \prod_{i=0}^{2} (n-i).$$

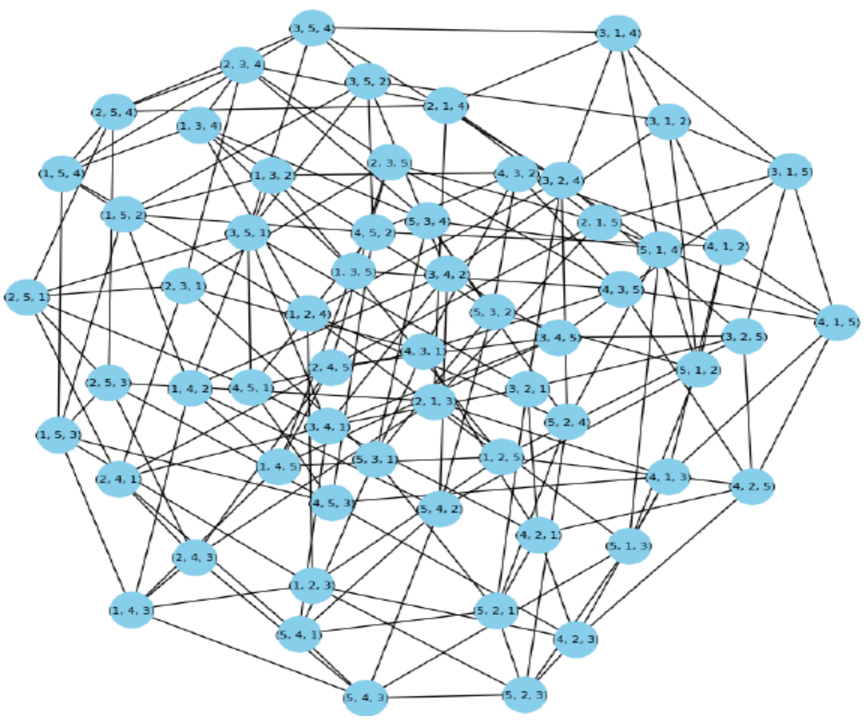

Figure 5: (5,3)-*arrangement graph*

Now, the $C_q\big(G_n(3)\big)$ code's minimum weight is examined and show that it equals $3(n-3)$ using induction on n.

Let's examine the base case $n = 5$. We show that $\omega\left(C_q\big(G_5(3)\big)\right) = 6$. The generator matrix of $A(5,3)$ is given by

$$\mathcal{G}_5(3) = \left(\begin{array}{c|c|c|c|c|c|c|c|c} \mathcal{G}_4(2) & 0 & 0 & 0 & 0 & \mathcal{I}_1 & 0' & 0'' & 0''' \\ \hline 0 & \mathcal{G}_4(2) & 0 & 0 & 0 & A_1' & \mathcal{I}_2 & 0'' & 0''' \\ \hline 0 & 0 & \mathcal{G}_4(2) & 0 & 0 & A_1'' & A_2' & \mathcal{I}_3 & 0''' \\ \hline 0 & 0 & 0 & \mathcal{G}_4(2) & 0 & A_1''' & A_2'' & A_3' & \mathcal{I}_4 \\ \hline 0 & 0 & 0 & 0 & \mathcal{G}_4(2) & A_1^{iv} & A_2''' & A_3'' & A_4' \end{array}\right).$$

To establish that $\omega\left(C_q(G_5(3))\right) = 6$, it is enough to show that every codeword $c \in C_q(G_5(3))$ satisfies $\omega(c) \geq 6$, and that equality holds for some codeword.

Let $c \in C_q(G_5(3))$. Divide the rows of $G_5(3)$ into blocks X_i, for $i \in Z_{1,5}$, and the columns into blocks C_i, for $i \in Z_{1,9}$. Write $c = (c_1, c_2, \ldots, c_9)$, where each component c_i is associated to the column block C_i. Therefore,

$$\omega(c) = \sum_{i=1}^{9} \omega(c_i)$$

yields the weight of c.

In order to examine c, we express it as an element of the span of r rows from a row block X_i, where $i \in Z_{1,5}$ and $r \in Z_{1,12}$.

Case 1

Subcase 1.1 Let c be created by linearly combining r rows from X_1. $wt(c) = 6$ if c is a row of $G_5(3)$. Otherwise, since $c_1 \in C_q(G_4(2))$, we have $\omega(c_1) \geq 4$. The component c_6 arises from I_1, so $\omega(c_6) = 2r$.

$Hence, \omega(c) = \omega(c_1) + \omega(c_6) \geq 4 + 2 = 6.$

For $r = 12$, we have $\omega(c_6) = 24$, and the inequality still holds.

Subcase 1.2 Let c be produced by linearly combining r rows from X_2. If c is a row, then $\omega(c) = 6$. Otherwise, since $c_2 \in C_q(G_4(2))$, $\omega(c_2) \geq 4$, and from the structure of $G_5(3)$, we have $\omega(c_6) + \omega(c_7) \geq 2$. Thus, $\omega(c) \geq 4 + 2 = 6$. The same argument applies to the blocks X_3, X_4, and X_5 by symmetry.

Case 2

If two distinct blocks, X_1 and X_2 are used to form c, then

$$\omega(c) \geq \omega(c_1) + \omega(c_2) \geq 4 + 4 = 8.$$

Thus, this bound is attained, and we have $\omega(c) \geq 6$ in all cases. Consequently,

$$\omega\left(C_q(G_5(3))\right) = 6.$$

Assume inductively that $\omega\left(C_q(G_{n-1}(3))\right) = 3\left((n-1)-3\right) = 3(n-4)$.

Now, we prove that $\omega\left(C_q(G_n(3))\right) = 3(n-3)$. Let $c = (c_1, c_2, \ldots, c_n, c_{n+1}, \ldots, c_{2n-1}) \in C_q(G_n(3))$. If c is a row of $G_n(3)$, then it is an generator vector of the graph. Since each row has exactly $3(n-3)$ non-zero entries. Hence, there exists a codeword with weight $3(n-3)$.

Now, let c be any codeword. If a single block X_i's rows are linearly combined to form c, then by Lemma 14, $c_i \in C_q(G_{n-1}(2))$, and thus $\omega(c_i) \geq 2(n-3)$. From the structure of $G_n(3)$, the remaining components contribute at least $n-3$, giving $\omega(c) \geq 3(n-3)$.

If c is formed using rows from multiple blocks, then each corresponding component contributes at least $2(n-3)$, and their contributions combine to yield $\omega(c) \geq 3(n-3)$.

Hence, each non-zero codeword has a minimum weight of $3(n-3)$ and equality is attained. Therefore,

$$\omega\left(C_q\left(G_n(3)\right)\right) = 3(n-3).$$

Thus, the code $C_q\left(G_n(3)\right)$ has length $\frac{3}{2}\prod_{i=0}^{3} \quad (n-i)$, dimension $\prod_{i=0}^{2} \quad (n-i)$, and minimum distance $3(n-3)$. Hence, it is a

$$\left[\frac{3}{2}\prod_{i=0}^{3} \quad (n-i),\ \prod_{i=0}^{2} \quad (n-i),\ 3(n-3)\right]_q$$

q-ary linear code. □

The $A(n,k)$ possess a high degree of symmetry. In view of Proposition 1, this symmetry leads to the following result concerning permutation decoding to the code $C_q\left(G_n(3)\right)$.

Theorem 17. Let S be the group of automorphisms of $A(n,3)$, $\forall\, n > 4$. Then any transitive subgroup of S constitutes a permutation decoding set that provides full error-correcting capability for the code $C_q\left(G_n(3)\right)$.

6 Construction of the generator matrix for $A(n,4)$

For every $n > 5$, we consider the codes obtained from the generator matrix of $A(n,4)$. This graph contains $\prod_{i=0}^{3} \quad (n-i)$ vertices and $2\prod_{i=0}^{4} \quad (n-i)$ edges. Hence, the associated generator matrix $G_n(4)$ has size $\prod_{i=0}^{3} \quad (n-i) \times 2\prod_{i=0}^{4} \quad (n-i)$.

The matrix $G_n(4)$ is represented by

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c}
\mathcal{G}_{n-1}(3) & 0 & 0 & \dots & 0 & 0 & 0 & \mathcal{I}_1 & 0' & 0'' & \dots & 0^{(n-3)} & 0^{(n-2)} \\ \hline
0 & \mathcal{G}_{n-1}(3) & 0 & \dots & 0 & 0 & 0 & \mathcal{B}'_1 & \mathcal{I}_2 & 0'' & \dots & 0^{(n-3)} & 0^{(n-2)} \\ \hline
0 & 0 & \mathcal{G}_{n-1}(3) & \dots & 0 & 0 & 0 & \mathcal{B}''_1 & \mathcal{B}'_2 & \mathcal{I}_3 & \dots & 0^{(n-3)} & 0^{(n-2)} \\ \hline
\vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \hline
0 & 0 & 0 & 0 & \mathcal{G}_{n-1}(3) & 0 & 0 & \mathcal{B}_1^{(n-3)} & \mathcal{B}_2^{(n-4)} & \mathcal{B}_3^{(n-5)} & \dots & \mathcal{I}_{n-2} & 0^{(n-2)} \\ \hline
0 & 0 & 0 & 0 & 0 & \mathcal{G}_{n-1}(3) & 0 & \mathcal{B}_1^{(n-2)} & \mathcal{B}_2^{(n-3)} & \mathcal{B}_3^{(n-4)} & \dots & \mathcal{B}'_{n-1} & \mathcal{I}_{n-1} \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & \mathcal{G}_{n-1}(3) & \mathcal{B}_1^{(n-1)} & \mathcal{B}_2^{(n-2)} & \mathcal{B}_3^{(n-3)} & \dots & \mathcal{B}''_{n-2} & \mathcal{B}'_{n-1}
\end{array}\right)$$

In this case, O is the zero matrix of size $\prod_{i=0}^{2} \quad (n-i-1) \times \frac{3}{2}\prod_{i=0}^{3} \quad (n-i-1)$. With dimensions $\prod_{i=0}^{2} \quad (n-i-1) \times (n-j)\prod_{i=2}^{4} \quad (n-i)$, the matrix $O^{(j-1)}$ denotes a zero matrix for $j \in Z_{2,n-1}$. Additionally, the block I_j has the same dimensions as $O^{(j-1)}$ for every $j \in Z_{2,n-1}$.

$$\mathcal{I}_1 = \begin{pmatrix} \underbrace{11\ldots1}_{(n-4)} & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & & \vdots \\ 0 & 0 & \underbrace{11\ldots1}_{(n-4)} & 0 & 0 \\ \vdots & & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \underbrace{11\ldots1}_{(n-4)} \end{pmatrix}.$$

In this matrix, each row of I_1 contains exactly $(n-4)$ ones placed along a diagonal pattern, and all other entries are zero. For I_j with $j \in Z_{2,n-1}$, each row contains at most $(n-4)$ ones, and the remaining entries are zero, arranged according to the block structure.

Let $m, l \in Z_{1,j-2}$ satisfy $m + l = j$. Then block $B_j^{(m)}$ is a matrix of size $\prod_{i=0}^{2} (n-i-1) \times (n-j)\prod_{i=2}^{4} (n-i)$. Moreover, each row of this matrix has at most one non-zero entry 1.

Copies of $G_{n-1}(3)$ are positioned along the diagonal in the first n column blocks. The matrices I_j, $B_j^{(m)}$, and $O^{(j-1)}$ are used to create the remaining $n-1$ column blocks.. These column blocks are represented by $C_1, C_2, \ldots, C_{2n-1}$.

Using the structure of $A(n,4)$, we obtain the following results, which describe the $|V|$ and $|E|$ of the graph and also the parameters of the code are determined.

Lemma 18. In $A(n,4)$, each row block of $G_n(4)$ contains a sub matrix which is the generator matrix of $A(n-1,3)$. Hence, for any codeword $c \in C_q(G_n(4))$, the component corresponding to a row block belongs to the code $C_q(G_{n-1}(3))$.

Theorem 19. For $n > 5$, $A(n,4)$s is a regular graph with $\prod_{i=0}^{3} (n-i)$ vertices and $2\prod_{i=0}^{4} (n-i)$ edges.

6.1 Fundamental Parameters of Error-Correcting Codes from $A(n,4)$

Theorem 20. Let $G_n(4)$ be $A(n,4)'$s generator matrix. The code $C_q(G_n(4))$ for any $n > 5$ and any odd prime q this matrix yields a q-ary linear code with parameters

$$\left[2\prod_{i=0}^{4} (n-i),\ \prod_{i=0}^{3} (n-i),\ 4(n-4)\right]_q.$$

Proof. We first examine the code length, which is $2\prod_{i=0}^{4} (n-i)$. Since $G_n(4)$ is the generator matrix corresponding to the arrangement graph $A(n,4)$, this graph contains $\prod_{i=0}^{3} (n-i)$ vertices and $2\prod_{i=0}^{4} (n-i)$ edges. Consequently, the generator matrix $G_n(4)$ has dimension

$$\prod_{i=0}^{3} (n-i) \times 2\prod_{i=0}^{4} (n-i).$$

It follows from the generator matrix $G_n(4)$ that the code $C_q(G_n(4))$ has length $2\prod_{i=0}^{4} (n-i)$.

Proposition 5 states that $A(n,4)$ is a connected graph. For $n > 5$, the cycle $p_1 \to p_2 \to p_3 \to p_1$ provides an odd-length closed path.

For example, the vertices $p_1 = (6,5,4,1)$, $p_2 = (6,5,4,2)$, and $p_3 = (6,5,4,3)$ form such a cycle in the $A(6,4)$, where each p_i belongs to $\{1,2,\dots,n\}$.

According to proposition 2, for any $n > 5$, the rank of $A(n,4)'$s generator matrix over F_q (for odd prime q) equals the number of vertices since $A(n,4)$ is connected and contains an odd cycle. Hence,

$$dim\left(C_q\left(G_n(4)\right)\right) = |V| \; in \; A(n,4)$$
$$= \prod_{i=0}^{3} \quad (n-i).$$

Next, we use induction on n to show that the least weight of $C_q\left(G_n(4)\right)$ is $4(n-4)$. The generator matrix $G_6(4)$ can be expressed in block form using copies of $G_5(3)$, arising from the recursive structure of $A(n,4)$. We now consider $G_6(4)$,

$$\mathcal{G}_6(4) = \left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
\mathcal{G}_5(3) & 0 & 0 & 0 & 0 & 0 & \mathcal{I}_1 & 0' & 0'' & 0''' & 0^{iv} \\ \hline
0 & \mathcal{G}_5(3) & 0 & 0 & 0 & 0 & \mathcal{B}_1' & \mathcal{I}_2 & 0'' & 0''' & 0^{iv} \\ \hline
0 & 0 & \mathcal{G}_5(3) & 0 & 0 & 0 & \mathcal{B}_1'' & \mathcal{B}_2' & \mathcal{I}_3 & 0''' & 0^{iv} \\ \hline
0 & 0 & 0 & \mathcal{G}_5(3) & 0 & 0 & \mathcal{B}_1''' & \mathcal{B}_2'' & \mathcal{B}_3' & \mathcal{I}_4 & 0^{iv} \\ \hline
0 & 0 & 0 & 0 & \mathcal{G}_5(3) & 0 & \mathcal{B}_1^{iv} & \mathcal{B}_2''' & \mathcal{B}_3'' & \mathcal{B}_4' & \mathcal{I}_5 \\ \hline
0 & 0 & 0 & 0 & 0 & \mathcal{G}_5(3) & \mathcal{B}_1^{v} & \mathcal{B}_2^{iv} & \mathcal{B}_3''' & \mathcal{B}_4'' & \mathcal{B}_5'
\end{array}\right).$$

To establish that the minimum weight of $C_q\left(G_6(4)\right)$ is 8, it suffices to verify two facts: (i) $\omega(c) \geq 8, \forall\, c \in C_q\left(G_6(4)\right)$, (ii) $\exists\, c \in C_q\left(G_6(4)\right)$ with $\omega(c) = 8$.

Let $c \in C_q\left(G_6(4)\right)$. Rows of $G_6(4)$ are partition into blocks X_i, for $i \in Z_{1,6}$, and the columns into blocks C_i, for $i \in Z_{1,11}$. Express $c = (c_1, c_2, \dots, c_{11})$, where each component c_i corresponds to the column block C_i. The weight of c can then be written as

$$\omega(c) = \sum_{i=1}^{11} \quad \omega(c_i).$$

We examine c by writing it as a sum of r rows from a fixed row block X_i over F_q, where $1 \in Z_{1,6}$ and $r \in Z_{1,60}$.

Case 1. Subcase 1.1. Assume that c is formed from r rows of X_1. If c itself is a single row of $G_6(4)$, then clearly $\omega(c) = 8$. Otherwise, since $c_1 \in C_q\left(G_5(3)\right)$, we have $\omega(c_1) \geq 6$. Combining r rows of the identity matrix I_1 yields the component c_7, with $\omega(c_7) = 2r$. Therefore,

$$\omega(c) = \omega(c_1) + \omega(c_7) \geq 6 +$$
$$= 8.$$

In the extreme case when $r = 60$, we have $\omega(c_7) = 120$, which clearly implies $\omega(c) \geq 8$.

Subcase 1.2. Let us now assume that c is derived from r rows of X_2. In a similar manner, if c lies in rows of $G_6(4)$, then $\omega(c) = 8$. Since $c_2 \in C_q\left(G_5(3)\right)$, we have $\omega(c_2) \geq 6$. From the block structure of $G_6(4)$, it follows that $\omega(c_7) + \omega(c_8) \geq 2$. Hence,

$$\omega(c) \geq \omega(c_2) + \omega(c_7) + \omega(c_8) \geq 6 + 2 = 8.$$

If $r = 60$, then $\omega(c_7) + \omega(c_8) = 120 \geq 8$, which again gives $\omega(c) \geq 8$.

By similar reasoning, the same conclusion holds when c is formed from rows of X_3, X_4, X_5, or X_6. Case 2. Let c is formed using r rows from X_1 together with r' rows from X_2. Given that $c_1, c_2 \in C_q(G_5(3))$, $\omega(c_1), \omega(c_2) \geq 6$ and therefore

$$\omega(c) \geq \omega(c_1) + \omega(c_2) \geq 12.$$

An identical argument applies when rows are taken from any two distinct row blocks X_i and X_j.

Combining all cases, we conclude that $\omega(c) \geq 8$ for every $c \in C_q(G_6(4))$. Since a row of $G_6(4)$ has weight 8, it follows that

$$\omega\left(C_q(G_6(4))\right) = 8.$$

To extend this result, we use the recursive block structure of $G_n(4)$, which involves copies of $G_{n-1}(3)$. By induction hypothesis, assume

$$\omega\left(C_q(G_{n-1}(4))\right) = 4((n-1)-4).$$

We now establish that $\omega\left(C_q(G_n(4))\right) = 4(n-4)$.

Let $c = (c_1, c_2, \dots, c_n, c_{n+1}, \dots, c_{2n-1}) \in C_q(G_n(4))$. If c is a row of $G_n(4)$, then it represents an incidence vector of $A(n,4)$. Since the graph is regular of degree $4(n-4)$, each vertex is adjacent to exactly $4(n-4)$ vertices, and hence $\omega(c) = 4(n-4)$. Thus, a codeword of this weight exists.

For every choice of c, we claim $\omega(c) \geq 4(n-4)$. Let c be expressed as a linear sum of k rows from a row block X_i. Then the component c_i belongs to $C_q(G_{n-1}(3))$, which implies that $\omega(c_i) \geq 3(n-4)$.

From the structure of $G_n(4)$, each selected row contributes non-zero entries to at least $n-4$ additional column blocks corresponding to identity submatrices. Hence,

$$\omega(c_{n+1}) + \cdots + \omega(c_{2n-1}) \geq n-4.$$

Thus, $\omega(c) \geq 3(n-4) + (n-4) = 4(n-4)$.

Similarly, when c is formed from rows of multiple row blocks, since each component contributes at least $3(n-4)$ to the weight. Consequently,

$$\omega(c) \geq 4(n-4).$$

Combining all cases, we conclude that

$$\omega\left(C_q(G_n(4))\right) = 4(n-4).$$

Consequently, $4(n-4)$ is the minimal distance of the code $C_q(G_n(4))$.

Thus, $C_q(G_n(4))$ generated by $G_n(4)$ has length $2\prod_{i=0}^{4}(n-i)$, dimension $\prod_{i=0}^{3}(n-i)$, and minimum distance $4(n-4)$. Therefore, $C_q(G_n(4))$ is a

$$\left[2\prod_{i=0}^{4}(n-i),\ \prod_{i=0}^{3}(n-i),\ 4(n-4)\right]_q$$

q-ary linear code. □

By proposition 1, The next theorem shows how the automorphism group of $A(n,4)$ can be used to obtain a PD-set for permutation decoding.

Theorem 21. Let S denote the automorphism group of $A(n,4)$. Then any subgroup of S acting transitively serves as a permutation decoding set for complete error correction of the code $C_q\big(G_n(4)\big)$.

The following theorem generalizes the parameters of the q-ary linear codes obtained from the generator matrix of arrangement graphs.

Theorem 22. Let $G_n(k)$ be the generator matrix of $A(n,k)$. Then, for any $n > k+1$ and any odd prime q, the code $C_q\big(G_n(2)\big)$ is a q-ary linear code with parameters

$$\left[\frac{k}{2}\prod_{i=0}^{k}(n-i),\ \prod_{i=0}^{k-1}(n-i),\ k(n-k)\right]_q.$$

7 Generating the codes from the Möbius–Kantor graph

7.1 Möbius–Kantor graph

The Möbius–Kantor graph arises as the Levi graph of the Möbius–Kantor configuration. This classical incidence of point-line configuration [10] consists of eight points and eight lines arranged so that each line is incident with precisely three points, while every point lies on exactly three lines. Let $V = U \cup W$ where $U = \{0,1,\dots,7\}$ and $W = \{8,9,\dots,15\}$ are disjoint. For each vertex $i \in U$, define its neighbourhood $N(i) \subset V$ by

$$N(i) = \{\,8+i,\ 8+(i+1\ mod \ldots + (i+3\ mod\ 8$$

where all additions are taken modulo 8. Because of symmetry, every vertex in W is incident with exactly three vertices in U. Consequently, the resulting construction forms a 3-regular bipartite graph on 16 vertices with 24 edges.

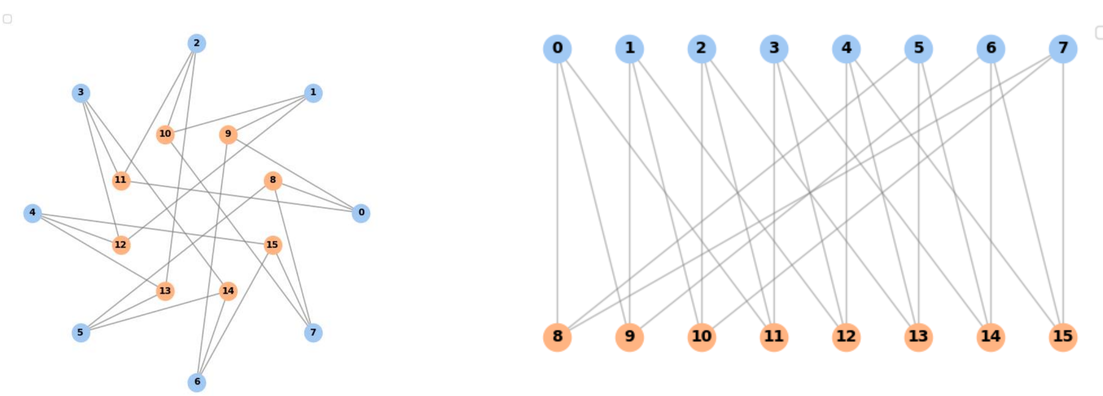

Figure 6: Möbius–Kantor graph.

The incidence matrix G_M of the Möbius–Kantor graph is constructed in a similar way. Its rows are divided into two blocks, denoted by X_1 and X_2, where X_1 represents the point vertices and X_2 represents the line vertices. The columns are arranged so that, for each point, the edges connecting it to all incident lines appear together. With this arrangement, we obtain

$$\mathcal{G}_M = \left(\begin{array}{cccccccccccccccccccccccc}
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
\hline
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0\\
0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&1&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&0&0
\end{array}\right)$$

Theorem 22. Let G_m denote the Möbius–Kantor graph. Let G_M be its incidence matrix. For any prime q, consider the code $C_q(G_m)$ defined as the row space of G_M over the finite field F_q. This code has parameters

$$[n,k,d]_q = [24,15,3]_q.$$

Proof. The $C_q(G_M)$ obtained from G_M, has length 24 because the graph contains 24 edges. Since the graph is connected and bipartite, proposition 2 implies that the dim($C_q(G_M)$)= $|V|-1=15$. A corresponding parity-check matrix, denoted by H_M, was then computed for this code.

$$H_M = \left(\begin{array}{cccccccccccccccccccccccc}
1&0&q-1&0&0&0&0&0&0&1&0&q-1&0&0&0&0&0&0&1&q-1&0&1&q-1&0\\
0&1&q-1&0&0&0&0&0&0&1&0&q-1&0&0&0&0&0&0&1&0&q-1&0&0&0\\
0&0&0&1&0&q-1&0&0&0&0&0&0&1&0&q-1&0&0&0&0&1&q-1&0&0&0\\
0&0&0&0&1&q-1&0&0&0&0&0&0&1&0&q-1&0&0&0&0&0&0&1&0&q-1\\
0&0&0&0&0&0&1&0&q-1&0&0&0&0&0&0&1&0&q-1&0&0&0&0&1&q-1\\
0&0&0&0&0&0&0&1&q-1&q-1&0&1&0&0&0&1&0&q-1&q-1&1&0&q-1&1&0\\
0&0&0&0&0&0&0&0&0&0&1&q-1&q-1&0&1&0&0&0&1&q-1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&q-1&q-1&0&1&0&0&0&1&q-1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&q-1&q-1&1&0&q-1&1&0
\end{array}\right)$$

Proposition 3 demonstrates that it is sufficient to find a group of three linearly dependent columns of H_M and then verify that no two columns satisfy the same requirement in order to calculate the code's minimum weight. Let c_i denotes i^{th} column of H_M. From H_M matrix we obtain the relation

$$c_3 + (q-1)c_1 + (q-1)c_2 = (0,$$

which shows that the three columns c_1, c_2, c_3 are linearly dependent. Consequently, the minimum distance satisfies $d\left(C_q(G_M)\right) \leq 3$.

Next, consider a possible dependence involving only two columns, say $\alpha_1(c_i) + \alpha_2(c_j) = 0$ where $\alpha_1, \alpha_2 \in F$. Writing this entry wise gives $\alpha_1(c_{1i}, c_{2i}, \dots, c_{9i}) + \alpha_2(c_{1j}, c_{2j}, \dots, c_{9j}) = 0$. Because of the structure of H_M, there exists a row index $k \in \{1,2,\dots,9\}$ for which exactly one of the entries c_{ki} or c_{kj} is non-zero.

Case 1: If $c_{ki} \neq 0$ and $c_{kj} = 0$ then the k^{th} coordinate gives $\alpha_1 c_{ki} = 0$, which implies $\alpha_1 = 0$. Since c_j is non-zero, we must also have $\alpha_2 = 0$.

Case 2: If $c_{ki} = 0$ and $c_{kj} \neq 0$ then the same argument yields $\alpha_2 = 0$ and hence $\alpha_1 = 0$. Therefore, the equation $\alpha_1(c_i) + \alpha_2(c_j) = 0$ admits only the trivial solution. This implies that any two columns are linearly independent in H_M . Consequently, we have $d\left(C_q(G_M)\right) \geq 3$. Combining this with the earlier inequality, we conclude that $d\left(C_q(G_M)\right) = 3$. Therefore, $[24,15,3]_q$ are the fundamental parameters of the code are produced by parity-check and generator matrix of the G_m. □

Since G_m is both vertex-transitive and edge-transitive, its group of automorphism provides a natural framework for constructing permutation decoding sets for the associated code. We get the following outcome by using Proposition 1.

Theorem 23. Let S denote the group of automorphism of G_m. Then any subgroup of S of order 96 with transitive action serves as a permutation decoding set, allowing the code produced by G_M to have complete error correction.

8 Conclusion

In this work, we develop linear error-correcting codes associated with the Möbius–Kantor graph. Furthermore, we construct linear codes derived from $G_n(k)$ of the $A(n,k)$ for all cases satisfying $n > k + 1$. We also carry out a detailed study of the particular graphs $A(n,2)$ for $n > 3$, $A(n,3)$ for $n > 4$, and $A(n,4)$ for $n > 5$, and compute the corresponding parameters of the constructed codes obtained from their generator matrices. The methods developed for these particular values of k can be extended in a similar way to obtain codes for general (n,k)-arrangement graphs whenever $n > k + 1$.

Thus, the constructions and results presented here provide a framework for generating codes from arrangement graphs of arbitrary dimension within this range. We also considered the special boundary case $n = k + 1$. In this situation, the structure of the graph changes, as it contains no odd cycles. By Proposition 1, it follows that dimension of associated code equivalently, the rank of incidence matrix is given by $|V| - 1$. A more comprehensive analysis of the codes arising in this setting will be addressed in a subsequent work.

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Conflicts of Interest

The author declare that they have no conflicts of interest regarding this publication.

AI Disclosure

The authors declare that no assistance is taken from generative AI to write this article

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