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An Analysis of Fundamental Parameters of Graph-Based Linear Error-Correcting Codes

X. Juliet Gokila[1], C. Durairajan[1*] and R. Theivaraman[2]

[1]Department of Mathematics, Bharathidasan University,

Tiruchirappalli - 620 024, Tamil Nadu, India.

[1]juliet6589111@gmail.com
[1*]cdurai66@bdu.ac.in.

[2]Department of Mathematics, SRM Valliammai Engineering College,

Kattankulathur – 603 203, Chennai, Tamil Nadu, India.

[2]deivaraman@gmail.com

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Abstract

In this work, we develop the linear codes generated by the incidence matrices of certain finite graphs over the finite field F_q. The main objective is to determine the fundamental parameters of the associated codes, namely the length, dimension, and minimum distance, and to examine the influence of graph structures on the resulting coding properties. Both bipartite and non-bipartite graphs are considered in this work. For the bipartite case, the Hypercube graph Q_n, Gray graph, and Nauru graph are examined, and the corresponding linear codes are shown to have parameters $[n2^{n-1}, 2^n - 1, 2]_q, [81, 53, 3]_q$, and $[36, 23, 3]_q$, respectively. Further, for the non-bipartite graphs $GP(5,2)$ and the Wagner graph, the associated linear codes are determined to have parameters $[15, 10, 3]_q$ and $[12, 8, 3]_q$, respectively.

1 Introduction

Coding theory is an important area of mathematics concerned with the reliable transmission and storage of information through noisy communication channels. Linear codes over finite fields have been extensively studied because of their algebraic structure and efficient error-correcting capability. The interaction between coding theory and graph theory has led to the development of several classes of graph-based codes constructed from adjacency and incidence matrices of graphs.

The study of graph-theoretic structures began with important contributions in combinatorics and algebraic graph theory. In 1947, Kagno [16] studied Desargues and Pappus graphs together with their automorphism groups. Later, Coxeter [3] investigated self-dual configurations and regular graphs in 1950, which contributed significantly to the theory of symmetric graphs. Folkman [9] further studied regular line-symmetric graphs in 1967, emphasizing the structural and symmetry properties of regular graphs.

The development of coding theory was greatly influenced by the work of MacWilliams [19] in 1964, where permutation decoding for systematic codes was introduced. This established an important connection between automorphism groups and decoding methods. Subsequently, MacWilliams and Sloane [20] presented a comprehensive treatment of error-correcting codes and their algebraic foundations. Important introductory and advanced texts on coding theory were later provided by Van Lint [26], Huffman and Pless [14], while Godsil and Royle [11] developed the algebraic aspects of graph theory relevant to coding applications.

The connection between graph theory and coding theory gained further importance through the study of incidence matrix codes. In 1992, Day and Tripathi [6] introduced arrangement graphs as a class of generalized star graphs and studied their combinatorial properties. Haemers, Peeters, and Van Rijckevorsel [13] investigated binary codes associated with strongly regular graphs in 1999, establishing important relationships between graph parameters and code properties.

Tonchev [25] studied error-correcting codes arising from graphs in 2002 and demonstrated the usefulness of graph-theoretic constructions in coding theory. Further developments were made by Key, Moori, and Rodrigues [18], who investigated codes associated with triangular graphs and permutation decoding. Their work emphasized the role of automorphism groups and transitive actions in decoding techniques.

In 2011, several important contributions were made in the study of codes generated from incidence matrices of graphs. Fish, Key, and Mwambene [8] studied linear codes constructed from the incidence matrices of graphs on 3-sets. They determined important code parameters such as the dimension and minimum distance and analyzed the automorphism groups of the associated graphs. The authors also investigated permutation decoding sets arising from these graph symmetries.

Ghinelli and Key [10] investigated linear codes obtained from the incidence matrices and line graphs of Paley graphs. Their work focused on determining the structural properties of the associated codes, including dimensions, dual codes, and minimum distances. They further studied how the strong regularity and symmetry of Paley graphs influence permutation decoding methods.

Key, Fish, and Mwambene [18] studied codes associated with Hamming graphs $H^k(n, 2)$ for $k \geq 2$. The authors analyzed the rank of the incidence matrices and obtained the parameters of the resulting linear codes. They also examined automorphism groups and constructed permutation decoding sets using the transitive properties of Hamming graphs.

In the same period, Fish, Kumwenda, and Mwambence [7] examined codes and combinatorial designs arising from triangular graphs and their line graphs. They determined the coding parameters of the associated incidence matrix codes and investigated the relationship between graph structures and combinatorial designs. The paper also studied the role of graph automorphisms in permutation decoding.

Dankelmann, Key, and Rodrigues [4] studied linear codes generated from the incidence matrices of finite graphs over finite fields. They established general results connecting graph-theoretic properties such as bipartiteness and connectivity with the dimension and minimum distance of the associated codes. Their work provided a systematic framework for studying graph-based incidence matrix codes.

Further advances were made by Dankelmann, Key, and Rodrigues [5], who characterized graphs using codes generated from their incidence matrices. The authors showed that certain graph properties can be completely determined from the corresponding incidence matrix codes. They also investigated equivalence conditions between graphs through the algebraic structure of the associated codes.

Polverino and Zullo investigated codes arising from incidence matrices of points and hyperplanes in projective geometries. Their work studied the dimensions and weight distributions of these codes and established connections between finite geometry and coding theory. The authors also analyzed geometric conditions influencing the structure of the resulting linear codes.

Martinez-Bernal, Valencia, and Villarreal [15] extended these ideas to signed graphs and studied the corresponding linear codes. They developed algebraic methods to construct codes from signed incidence matrices and investigated their generator matrices, dimensions, and structural properties. The paper demonstrated how signed graph theory can be applied in coding-theoretic constructions.

Recently, Annamalai and Durairajan [1] studied linear codes generated from the incidence matrices of unit graphs. They determined the parameters of the associated codes and analyzed how the algebraic properties of unit graphs influence the resulting linear codes. The work also examined the relationship between graph connectivity and coding parameters.

Saranya and Durairajan [23] investigated codes associated with arrangement graphs, particularly $(n,1)$-arrangement graphs and $(n,2)$-arrangement graphs. They computed the dimensions and minimum distances of the corresponding incidence matrix codes and studied the influence of arrangement graph structures on code properties. Their work further contributed to the study of graph-based linear code constructions.

Motivated by these developments, the present work investigates linear codes generated by the incidence matrices of selected finite graphs over the finite field F_q. The primary objective is to determine the basic parameters of the associated codes, namely the length, dimension, and minimum distance, and to analyze the influence of graph structures on these parameters.

Both bipartite and non-bipartite graphs are considered in this work. For the bipartite case, the Hypercube graph Q_n, Gray graph, and Nauru graph are studied, and the corresponding linear codes are shown to have parameters $[n2^{n-1}, 2^n - 1, n]_q, [81, 53, 3]_q$, and $[36, 23, 3]_q$, respectively. Further, for the non-bipartite graphs $GP(5,2)$ and the Wagner graph, the associated linear codes are determined to have parameters $[15, 10, 3]_q$ and $[12, 8, 3]_q$, respectively.

2 Preliminary

Let F_q be a finite field consisting of q elements. The vector space F_q^n then has dimension n over F_q. For a vector $x = (x_1, x_2, \dots, x_n) \in F_q^n$, the Hamming weight $w(x)$ of x is defined as the total number of non zero components in x. Explicitly,

$$w(x) = \sum_{j=1}^{n} w(x_j), \quad where \quad w(x_j) = \{1, \quad if\ x_j \neq 0, 0, \quad if\ x_j = 0.$$

Similarly, the Hamming distance between two vectors $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n) \in F_q^n$ is defined as the number of coordinates in which they differ. This distance is given by $d(x, y) = \sum_{j=1}^{n} w(x_j - y_j)$.

A nonempty subset $C$of F_q^n is called a q-ary code of length n, and the elements of C are known as codewords. If C forms a vector subspace of F_q^n over the finite field F_q, then C is referred to as q-ary linear code.

The minimum Hamming distance of a code C is defined by

$$d(C) = min\,\{d(c_1, c_2) \mid c_1, c_2 \in C,\ c_1 \neq c_2\},$$

where $d(c_1, c_2)$ denotes the Hamming distance between the codewords c_1 and c_2. The minimum weight of C, denoted by $w(C)$, is the least Hamming weight among all nonzero codewords in C. For a q-ary linear code, the minimum distance is equal to its minimum weight, that is, $d(C) = w(C)$.
In this paper, we consider only linear codes. A q-ary linear code with length n, dimension k, and minimum distance $d$is denoted by $[n, k, d]_q$.

For a linear code C, a generator matrix G is defined as a matrix whose rows constitute a basis of C. The code generated by G over the finite field F_q is denoted by $C_q(G)$. The dimension of this code is determined by the rank of G when considered over F_q. In particular, when the rows of G form a basis for C, the dimension of the code coincides exactly with the rank of the generator matrix. In [8], the author proved the following.

Proposition 1. Let $\Gamma = (V, E)$ be a regular graph of degree k with automorphism group A transitive on edges. Let M be an incidence matrix for Γ. If for a prime q, $C = C_q(M) = [|E|, |V| - \epsilon, k]_q$, where $\epsilon \in \{0,1, \dots, |V| - 1\}$, then any transitive subgroup of A will serve as a PD-set for full error correction for C.

Throughout this work, we focus on undirected, simple, and regular graphs. Let $\Gamma = (V, E)$ denote a graph, where V represents the vertex set and E denotes the edge set. For any two vertices $x, y \in V$, the notation $[x, y]$ is used to indicate an edge joining x and y The incidence matrix associated with Γ is defined as a $|V| \times |E|$ matrix $I = [a_{ij}]$, in which each row corresponds to a vertex and each column corresponds to an edge. The entries of this matrix are given by

$$a_{ij} = \{1, \quad if\ \ e_j\ is\ incident\ with\ the\ \ v_i, 0,$$

The following result was established in [4] and will be employed later in this work.

Proposition 2. Consider a connected graph $\Gamma = (V, E)$ with incidence matrix I. Let $C_q(I)$ denote the code generated by the rows of I over F_q. Then

$$dim\left(C_2(I)\right) = |V| - 1,$$

and for odd prime q,

$$dim\left(C_q(I)\right) = \{|V|,\ \ if\ \Gamma\ is\ not\ bipartite, |V| - 1,\ \ if\ \Gamma\ is\ bipartite.$$

Proposition 3. [22] Let C be a linear code and let H be a parity check matrix for C. Then the following statements are equivalent:

1. C has distance d.
2. Any $d-1$ columns of H are linearly independent and H has d columns that are linearly dependent.

3 Fundamental parameters in linear error-correcting codes from regular bipartite graphs

In this section, we determine the basic parameters such as the length, dimension, and minimum distance of the linear codes generated by the incidence matrices of certain bipartite graphs, namely the Hypercube graph Q_n, Gray graph, and Nauru graph. The corresponding code parameters are obtained as $[n2^{n-1}, 2^n - 1, n]_q, [81, 53, 3]_q$ and $[36, 23, 3]_q$ respectively.

3.1 Hypercube Graph Q_n

The hypercube graph Q_n is a well-known family of graphs consisting of all binary strings of length n. It is a connected, n-regular, bipartite, and symmetric graph with 2^n vertices and $n2^{n-1}$ edges.

Let $V(Q_n) = \{0,1\}^n$.

We partition the vertex set into two disjoint sets:

$$V_1 = \{x \in V(Q_n): sum\ of\ coordinates\ of\ x\ is\ even\},$$

and

$$V_2 = \{x \in V(Q_n): sum\ of\ coordinates\ of\ x\ is\ odd\}.$$

Then $V(Q_n) = V_1 \cup V_2, \quad V_1 \cap V_2 = \varnothing.$

For each vertex $x = (x_1, x_2, \dots, x_n) \in V(Q_n)$, define its neighbourhood by $N(x) = \{x^{(1)}, x^{(2)}, \dots, x^{(n)}\}$, where $x^{(i)}$ is obtained from x by changing the i^{th} coordinate.

Thus every vertex is adjacent to exactly n vertices, and hence the graph is n-regular. Since changing one coordinate changes the parity of the sum of coordinates, every edge joins a vertex in V_1 to a vertex in V_2. Hence Q_n is bipartite.

3.1.1 Construction of Incidence Matrix of Q_n

The incidence matrix of the hypercube graph Q_n can be constructed recursively from the incidence matrix of Q_{n-1}. The recursive construction follows from

$$Q_n = Q_{n-1} \square K_2,$$

where $\square$ denotes the Cartesian product of graphs and K_2 is the complete graph on two vertices. This means that the graph Q_n is obtained by taking two copies of the graph Q_{n-1} and joining corresponding vertices by additional edges.

Hence the incidence matrix of Q_n takes the recursive block form

$$\mathcal{G}(Q_n) = \left(\begin{array}{c|c|c} \mathcal{G}(Q_{n-1}) & 0 & I_{2^{n-1}} \\ \hline 0 & \mathcal{G}(Q_{n-1}) & I_{2^{n-1}} \end{array}\right),$$

Figure 1: Incidence matrix of Q_n

The upper block row corresponds to the vertices of the first copy $Q_{n-1}^{(1)}$, while the lower block row corresponds to the vertices of the second copy $Q_{n-1}^{(2)}$. Similarly,

- the first block column represents the edges of the first copy,
- the second block column represents the edges of the second copy,
- the third block column represents the matching edges between corresponding vertices.

For example, $Q_3 = Q_2 \square K_2$,

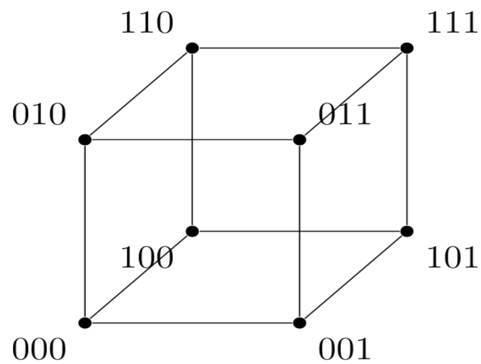

Figure 2: Q_3

so the graph Q_3 is obtained from two copies of the square graph Q_2 by joining corresponding vertices. Consequently,

$$\mathcal{G}(Q_3) = \left(\begin{array}{c|c|c} \mathcal{G}(Q_2) & 0 & I_4 \\ \hline 0 & \mathcal{G}(Q_2) & I_4 \end{array}\right) = \left(\begin{array}{cccc|cccc|cccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Figure 3: Incidence matrix of Q_3

Theorem 4. Let Q_n denote the hypercube graph of dimension n, and let $G(Q_n)$ be its incidence matrix. For any prime q, consider the linear code $C_q(Q_n)$ defined as the row space of $G(Q_n)$ over the finite field F_q. Then the code has parameters $[n2^{n-1}, 2^n - 1, n]_q$.

Proof. The hypercube graph Q_n contains 2^n vertices and $n2^{n-1}$ edges. Since the columns of the incidence matrix correspond to the edges of the graph, the code length $C_q(Q_n)$ is $n2^{n-1}$.

Let $C_q(Q_n)$ be the linear code generated by the incidence matrix $G(Q_n)$ over the finite field F_q. Then Proposition 2, $C_q(Q_n) = Row\ space\ of\ G(Q_n)$.

Since Q_n is connected, the rank of its incidence matrix is

$$rank\big(G(Q_n)\big) = |V(Q_n)| - 1 = 2^n - 1.$$

Therefore the dimension of the code is

$$dim\left(C_q(Q_n)\right) = 2^n - 1.$$

We now determine the minimum distance of the code generated by the incidence matrix of the hypercube graph Q_n by induction on n.

Base case:

Consider the hypercube graph Q_2. The graph Q_2 is a cycle of length four, and each vertex has degree 2. Hence every row of the incidence matrix $G(Q_2)$ contains exactly two non-zero entries.

Therefore every row of $G(Q_2)$ has Hamming weight 2. Since the rows generate the code $C_q(Q_2)$, there exists a non-zero codeword of weight 2. Thus $d\left(C_q(Q_2)\right) \leq 2$.

Now let x be a non-zero codeword of $C_q(Q_2)$. Since each row of the incidence matrix has weight 2, no non-zero linear combination of rows can produce a codeword of weight less than 2. Hence $d\left(C_q(Q_2)\right) \geq 2$.

Consequently,

$$d\left(C_q(Q_2)\right) = 2.$$

Inductive hypothesis:

Assume that for the hypercube graph Q_{n-1}, the linear code generated by its incidence matrix has minimum distance

$$d\left(C_q(Q_{n-1})\right) = n - 1.$$

Inductive step:

The hypercube graph Q_n is obtained recursively from two copies of Q_{n-1} by joining corresponding vertices. That is,

$$Q_n = Q_{n-1} \square K_2.$$

Hence the incidence matrix of Q_n admits the recursive block form

$$\mathcal{G}(Q_n) = \left(\begin{array}{c|c|c} \mathcal{G}(Q_{n-1}) & 0 & I_{2^{n-1}} \\ \hline 0 & \mathcal{G}(Q_{n-1}) & I_{2^{n-1}} \end{array}\right),$$

where $I_{2^{n-1}}$ is the identity matrix of order 2^{n-1}. Since every vertex of Q_n is adjacent to exactly one additional vertex compared with Q_{n-1}, each row of $G(Q_n)$ contains exactly one more non-zero entry than the corresponding row of $G(Q_{n-1})$.

By the induction hypothesis, each non-zero codeword arising from $G(Q_{n-1})$ has weight at least $n-1$. The additional identity block contributes exactly one further non-zero entry to each row. Therefore every non-zero row of $G(Q_n)$ has weight

$$(n-1)+1=n.$$

Hence there exists a non-zero codeword of weight n, which implies

$$d\left(C_q(Q_n)\right)\leq n.$$

Now consider any non-zero linear combination of rows of $G(Q_n)$. By the induction hypothesis, the contribution from the $G(Q_{n-1})$ blocks has weight at least $n-1$.

Further, the identity block $I_{2^{n-1}}$ has an additional non-zero coordinate corresponding to the connecting edge between the two copies of Q_{n-1}.

Therefore no non-zero linear combination can produce a codeword of weight less than n.

Thus $d\left(C_q(Q_n)\right)\geq n$.

Combining the inequalities,

$$d\left(C_q(Q_n)\right)=n.$$

Hence, by induction, $d\left(C_q(Q_n)\right)=n$ for all $n\geq 2$.

Hence the code parameters are

$$[n,k,d]_q=[n2^{n-1},\,2^n-1,\,n].$$

□

Theorem 5. Let $C_q(Q_n)$ be the linear code generated by the incidence matrix of the hypercube graph Q_n. Since the hypercube graph is edge-transitive, suitable edge-transitive subgroups of the automorphism group $Aut(Q_n)\cong Z_2^n\rtimes S_n$ can be used to construct PD-sets for the error correcting code $C_q(Q_n)$.

3.2 Gray graph

The Gray graph arises as the Levi graph of the Gray configuration. This classical incidence configuration consists of twenty seven points and twenty seven lines arranged so that each line is incident with exactly three points and each point lies on exactly three lines. Let $V=U\cup W$ where $V=\{0,1,\dots,26\}$ and $W=\{27,28,\dots,53\}$ are disjoint. For each vertex $i\in U$, define its neighbourhood $N(i)\subset W$ by

$$N(i)=\{\,27+i,\ 27+\left((i+1)\ mod\ 27\right),\ 27+\left((i+9)\ mod\ 27\right)\},$$

where all additions are taken modulo 27. By symmetry, each vertex of W is adjacent to exactly three vertices of U. Hence this construction yields a 3-regular bipartite graph with 54 vertices and 81 edges.

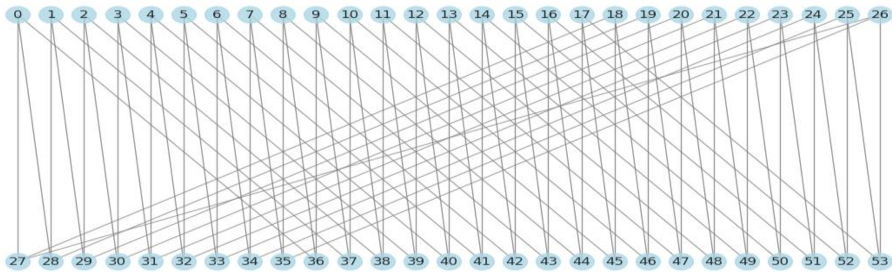

Figure 4: Gray graph.

We construct the incidence matrix of the Gray graph in an analogous manner. The rows are partitioned into two blocks, X_1 and X_2, where X_1 corresponds to the point vertices and X_2 corresponds to the line vertices. The columns are ordered so that, for each point, we group together the edges joining that point to all lines incident with it. Under this ordering, we obtain G_G of the Gray graph as

$$\left(\begin{array}{*{81}{c}}
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
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\end{array}\right)$$

Theorem 6. Let G_g denote the Gray graph. Let G_G be its incidence matrix. For any prime q, consider the linear code $C_q(G_g)$ defined as the row space of G_G over the finite field F_q. This code has parameters

$$[n,k,d]_q = [81,53,3]_q.$$

Proof. The linear code $C_q(G_g)$ is constructed from the G_G of the Gray graph. Since the Gray graph has 81 edges, the code length is 81. Because the graph is connected and bipartite, proposition 2 implies that the dimension of the code is $|V|-1$. The Gray graph contains 54 vertices, and therefore $dim\left(C_q(G_G)\right)=|V|-1=54-1=53$. A corresponding parity-check matrix for this code, denoted by H_G, was then determined.

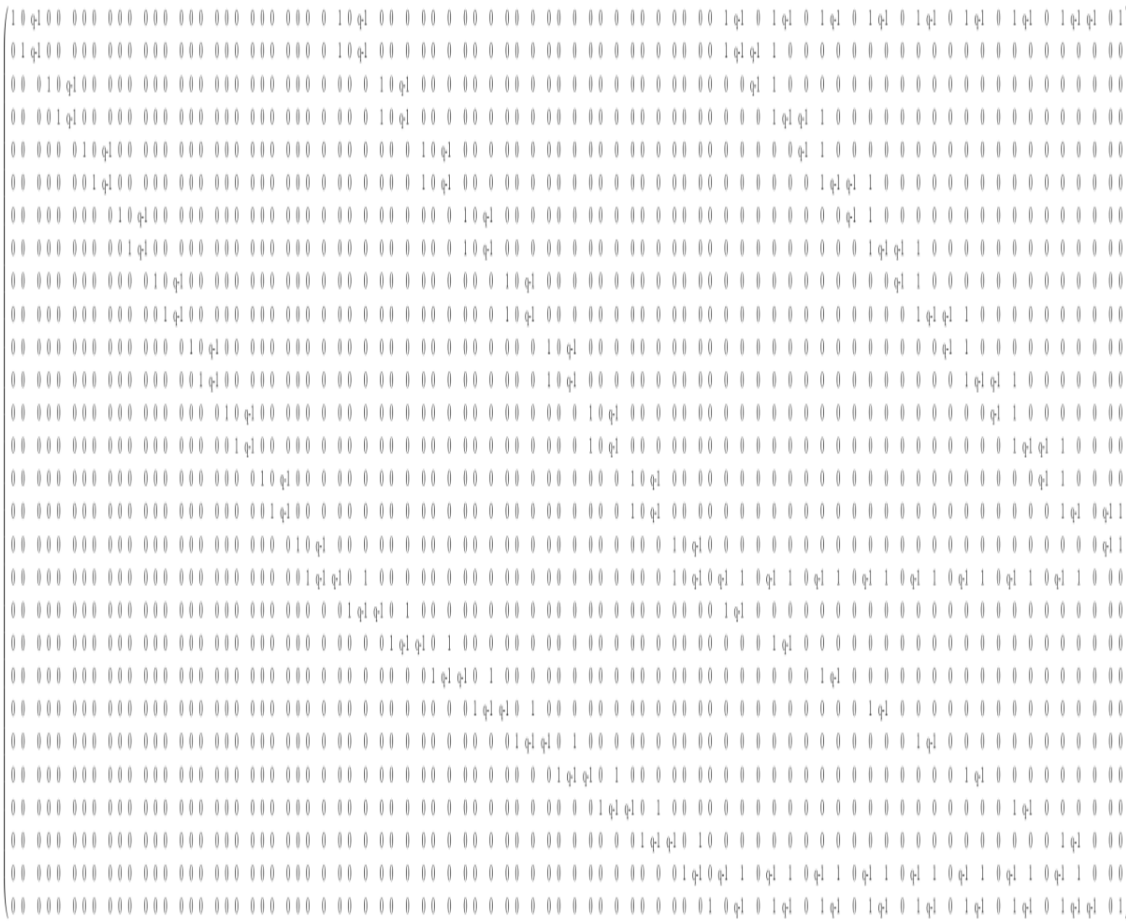

Figure 5: Parity check matrix of Gray graph.

To determine the minimum code weight, Proposition 3 shows that it is enough to find three columns of H_G that are linearly dependent and then verify that no pair of columns satisfies such a relation. Let c_i denote the i^{th} column of H_G. From the structure of the matrix we obtain

$$c_1+(q-1)c_2+(q-1)c_3=(0,0,\dots,0),$$

which shows that the columns c_1, c_2, c_3 are linearly dependent. Hence the minimum distance satisfies $d\left(C_q(G_G)\right) \leq 3$. Next, suppose that there exists a dependence involving only two columns, say $\alpha c_i + \beta c_j = 0$, where $\alpha, \beta \in F_q$. Writing this equality entrywise gives $\alpha(c_{1i}, c_{2i}, \dots, c_{21i}) + \beta(c_{1j}, c_{2j}, \dots, c_{21j}) = (0,0,\dots,0)$. Because of the structure of H_G, there exists a row index $k \in \{1,2,\dots,21\}$ for which exactly one of the entries c_{ki} or c_{kj} is non zero.

Case 1: If $c_{ki} \neq 0$ and $c_{kj} = 0$, then the k^{th} coordinate gives $\alpha c_{ki} = 0$, which implies $\alpha = 0$. Since c_j is non zero, we must also have $\beta = 0$.

Case 2: If $c_{ki} = 0$ and $c_{kj} \neq 0$, a similar argument gives $\beta = 0$ and hence $\alpha = 0$.

Thus the only solution to $\alpha c_i + \beta c_j = 0$ is the trivial one, and therefore any two columns of H_G are linearly independent. Consequently, $d\left(C_q(G_G)\right) \geq 3$.

Combining the two inequalities, we conclude that $d\left(C_q(G_G)\right) = 3$.

Hence the code generated by the incidence matrix of the Gray graph has parameters $[81,53,3]_q$.

Since the Gray graph is edge-transitive, its automorphism group can be used to construct permutation decoding sets for the associated code. Applying proposition 1, we obtain the following result.

Theorem 7. Let $C_q(G_G)$ be the linear code generated by the G_G of the Gray graph. Since the Gray graph is edge-transitive, suitable edge-transitive subgroups of the automorphism group $Aut(G_G)$ of order 1296 can be used to construct PD-sets for the code $C_q(G_G)$.

3.3 Nauru graph

N = + − +

The Nauru graph is one of the well-known members of the family of generalized Petersen graphs and is represented by $GP(12,5)$. It is a connected cubic bipartite symmetric graph consisting of twenty four vertices and thirty six edges. Since the graph is bipartite and 3-regular, it may also be viewed as the incidence graph of an abstract (12_3)-configuration. Let $V(G) = A \cup B$,

where $A = \{0,1,2,\dots,11\}$ and $B = \{12,13,14,\dots,23\}$ are disjoint vertex sets corresponding respectively to the outer cycle and the inner star polygon of the generalized Petersen construction.

For each vertex $i \in A$, define its neighbourhood by

where all arithmetic is performed modulo 12.

Similarly, for each vertex $(12+i) \in B$,

$$N(12+i) = \{12 + ((i+5)\ mod\ 12),\ 12 + ((i-5)\ mod\ 12),\ i\},$$

define

for $i \in \{0,1,\dots,11\}$.

Thus every vertex is adjacent to exactly three vertices, and consequently the graph is a □ cubic graph. Since the graph admits a bipartition into the sets A and B, it follows that the graph is bipartite. Moreover, the graph contains no odd cycles and has girth six.

The automorphism group of the Nauru graph acts transitively on both vertices and edges, establishing the graph as a symmetric graph. Owing to these structural properties, the graph has applications in the study of graph automorphisms, incidence structures, and linear code constructions.

3.3.1 Construction of Incidence matrix of the Nauru graph

Let us consider the Nauru graph,

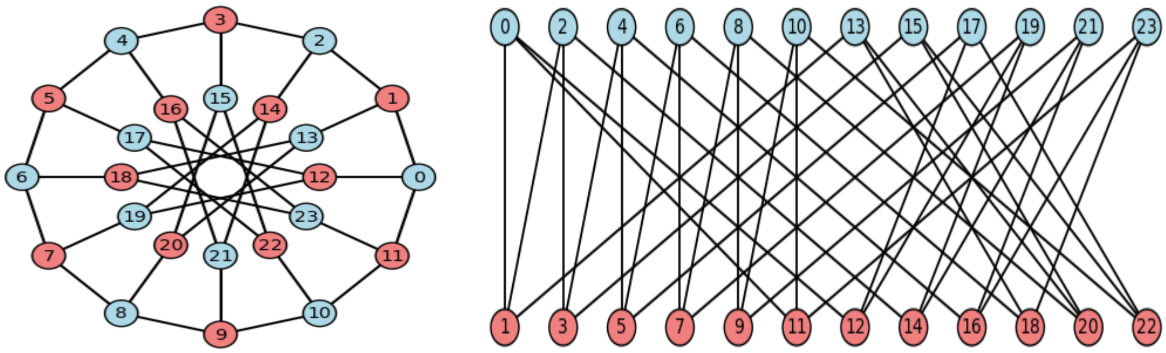

Figure 6: Nauru graph.

Since the Nauru graph is bipartite, the bipartition is defined as $V_1 = \{0,2,4,6,8,10,13,15,17,19,21,23\}$ and $V_2 = \{1,3,5,7,9,11,12,14,16,18,20,22\}$. Then $V(G) = V_1 \cup V_2$ with $V_1 \cap V_2 = \phi$.

From the construction of the graph, every edge connects a vertex of V_1 to a vertex of V_2. Indeed,

1. each outer cycle edge joins vertices of opposite parity,
2. each spoke edge connects a vertices from one partite set to another,
3. each inner edge also connects vertices belonging to opposite parity classes, since the step size 5 is odd.

Hence there are no edges joining two vertices within the same partite set. Therefore the Nauru graph is bipartite. The construction of the incidence matrix of the Nauru graph G_N

$$\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1
\end{pmatrix}$$

Figure 7: Incidence matrix of Nauru graph.

Theorem 8. Let N denote the Nauru graph and let G_N be its incidence matrix. For any prime q, consider the linear code $C_q(N)$ defined as the row space of G_N over the finite field F_q. Then the code $C_q(N)$ has parameters

$$[n,k,d]_q = [36,23,3]_q.$$

Proof. The linear code $C_q(N)$ is generated by G_N of Nauru graph. Since the Nauru graph contains 36 edges implies, the code length is 36.

The Nauru graph is connected and bipartite with 24 vertices. Hence, by Proposition 2, the rank of its G_N over the finite field F_q is

$$rank(G_N) = |V| - 1 = 24 - 1 \\ = 23.$$

Therefore,

$$k = dim\left(C_q(N)\right) = 23.$$

A corresponding parity-check matrix for the code, denoted by H_N, after permutating the columns of the matrix is obtained in the standard form

$$\left(\begin{array}{ccccccccccccccccccccccc|cccccccccccc}
q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q-1 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q-1 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & q-1 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q-1 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q-1 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q-1 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q-1 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & q-1 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q-1 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q-1 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q-1 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q-1 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right)$$

Figure 8: Parity-check matrix of Nauru graph.

$$H_N = (-A^T |\, I_{13}\,),$$

where A^T denotes the transpose of the matrix A and I_{13} is the identity matrix of order 13.

To determine the minimum distance of the code, Proposition 3 implies that it is sufficient to find three linearly dependent columns of H_N and then verify that no pair of columns is linearly dependent.

Let c_i denote the i^{th} column of H_N. From the structure of the parity-check matrix, we obtain

$$c_1 + (q-1)c_2 + (q-1)c_3 = (0,0,\dots,0).$$

Hence the columns c_1, c_2, and c_3 are linearly dependent. Therefore,

$$d\left(C_q(N)\right) \le 3.$$

Next, suppose that there exists a dependence relation involving only two columns of H_N. Then there exist $\lambda, \mu \in F_q$ such that $\lambda c_i + \mu c_j = 0$. Writing this relation coordinatewise gives

$$\lambda(c_{1i}, c_{2i}, \dots, c_{13i}) + \mu\left(c_{1j}, c_{2j}, \dots, c_{13j}\right) = (0,0,\dots,0).$$

Because of the structure of H_N, there exists a row index $t \in \{1,2,\dots,13\}$ for which exactly one of the entries c_{ti} or c_{tj} is non-zero.

Case 1: If $c_{ti} \neq 0$ and $c_{tj} = 0$, then the t^{th} coordinate gives $\lambda c_{ti} = 0$. Since $c_{ti} \neq 0$, it follows that $\lambda = 0$. Consequently, $\mu = 0$.

Case 2: If $c_{ti} = 0$ and $c_{tj} \neq 0$, a similar argument gives $\mu = 0$, and hence $\lambda = 0$.

Thus the only solution of $\lambda c_i + \mu c_j = 0$ is the trivial solution $\lambda = \mu = 0$.

Therefore any two columns of H_N are linearly independent. Consequently, $d\left(C_q(N)\right) \ge 3$.

Combining the inequalities,

$$d\left(C_q(N)\right) = 3.$$

Hence the linear code generated by the G_N of the Nauru graph has parameters

$$[36,23,3]_q.$$

Since the Nauru graph is edge-transitive, its automorphism group can be used in the construction of permutation decoding sets for the associated code. Applying Proposition 1, we obtain the following result.

Theorem 9. Let $C_q(N)$ be the linear code generated by the G_N of the Nauru graph N. Since the Nauru graph is edge-transitive, suitable edge-transitive subgroups of the automorphism group $Aut(N)$ of order 144 can be used to construct PD-sets for the code $C_q(N)$.

4 Fundamental parameters in linear error-correcting codes from regular non-bipartite graphs

In this section, we study the linear codes obtained from the incidence matrices of the non-bipartite graphs $GP(5,2)$ and the Wagner graph. The basic code parameters such as the length, dimension, and minimum distance are determined. The corresponding codes have parameters $[15, 10, 3]_q$ and $[12, 8, 3]_q$ respectively.

4.1 Generalized Petersen Graphs and Associated Linear Codes

For natural numbers n and k with $1 \le k \le \frac{n}{2}$, the generalized Petersen graph $GP(n,k)$ is defined as a graph whose vertex set is $\{u_i, v_i \mid i = 1,2,\dots,n\}$ and whose edge set consists of the edges $u_i u_{i+1}$, $u_i v_i$, and $v_i v_{i+k}$ for $i = \{1,2,\dots,n\}$, where the indices are taken modulo n. The generalized Petersen graph $P(n,k)$ has $2n$ vertices and $3n$ edges. It is not bipartite when n is odd, or when both n and k are even. However, the graph is bipartite when n is even and k is odd. The linear code $C_q\big(GP(n,k)\big)$ generated by the incidence matrix of $GP(n,k)$ has parameters $[\,3n,\,2n,\,3\,]$ when the graph is not bipartite. When the graph is bipartite, the code has parameters $[\,3n,\,2n-1,\,3\,]$.

4.1.1 Petersen graph PG(5,2)

In particular, by taking $n = 5$ and $k = 2$, we obtain the graph $GP(5,2)$, which is commonly called the Petersen graph. The graph has 10 vertices and 15 edges, and each vertex has degree 3. Since the graph contains odd cycles, it is non-bipartite. The graph consists of an outer cycle, an inner star-shaped structure, and spokes joining corresponding vertices of the two parts.

Let $V(P) = \{0,1,2,3,4,5,6,7,8,9\}$, where the vertices

$u_i = i, \qquad i = 0,1,2,3,4,$ represent the outer vertices and

$v_i = i + 5, \qquad i = 0,1,2,3,4,$ represent the inner vertices.

The edge set of the Petersen graph consists of the following three types of edges:

Outer cycle edges

$u_i u_{i+1}, \qquad i = 0,1,2,3,4,$ where the indices are taken modulo 5.

Spoke edges

$u_i v_i, \qquad i = 0,1,2,3,4.$

Inner star edges

$v_i v_{i+2}, \qquad i = 0,1,2,3,4,$ where the indices are taken modulo 5.

Using the above labeling, the edge set can be written explicitly as

$E(P)$
$= \{\,(0,1),(1,2),(2,3),(3,4),(4,0), \qquad (0,5),(1,6),(2,7),(3,8),(4,9), \qquad (5,7),(6,8),(7,9),(8$

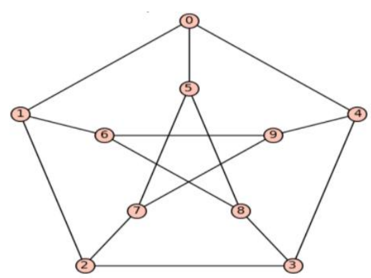

Figure 9: Petersen graph.

The generator matrix G_P, which is obtained from the G_P of the Petersen graph, is constructed by considering the relationship between the vertices and edges of the graph.

$$\left(\begin{array}{ccccc|ccccc|ccccc}
1&0&0&0&1&1&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&1&1&0&0&0&0&1&0&0&0&0&0\\
\hline
0&0&0&0&0&1&0&0&0&0&1&0&0&1&0\\
0&0&0&0&0&0&1&0&0&0&0&1&0&0&1\\
0&0&0&0&0&0&0&1&0&0&1&0&1&0&0\\
0&0&0&0&0&0&0&0&1&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&1&0&0&1&0&1
\end{array}\right)$$

Figure 10: Generator matrix of Petersen graph.

Theorem 10. Let q be a prime integer, and let $C_q(G_P)$ be the linear code over the finite field F_q generated by the incidence matrix of the Petersen graph G_P. Then the code $C_q(G_P)$ has parameters $[15, 10, 3]_q$.

Proof. The linear code $C_q(G_P)$, obtained from the G_P of the Petersen graph, has length 15 because the graph contains 15 edges. Since the Petersen graph is connected and non-bipartite, proposition 2 implies that the dimension of $C_q(G_P)$ is $|V| = 10$. Thus the code has dimension 10. A corresponding parity-check matrix, denoted by H_P, was then computed for this code.

$$\left(\begin{array}{cccccccccc|ccccc}
0 & 0 & 1 & q-1 & 1 & q-1 & 0 & q-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 1 & q-1 & 0 & q-1 & 0 & q-1 & 0 & 0 & 1 & 0 & 0 & 0\\
q-1 & 1 & 0 & 0 & 1 & 0 & 0 & q-1 & 0 & q-1 & 0 & 0 & 1 & 0 & 0\\
1 & q-1 & 1 & 0 & 0 & q-1 & 0 & 0 & q-1 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 1 & q-1 & 1 & 0 & 0 & q-1 & 0 & 0 & q-1 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

Figure 11: Parity-check matrix of Petersen graph.

To determine the minimum weight of the code, proposition 3 tells us that it is enough to find a set of three columns of H_P that are linearly dependent and then verify that no pair of columns has this property.

Let c_i denote the i^{th} column of H_P. From the matrix we obtain the relation

$$c_1 + c_2 + c_3 = (0,0,0,0,0),$$

for the binary field, and more generally,

$$c_3 + (q-1)c_1 + (q-1)c_2 = (0,0,0,0,0),$$

over F_q. Hence the three columns c_1, c_2, c_3 are linearly dependent. Therefore,

$$d\left(C_q(G_P)\right) \leq 3.$$

Next, suppose there exists a linear dependence involving only two columns, say

$$\alpha_1(c_i) + \alpha_2(c_j) = 0,$$

where $\alpha_1, \alpha_2 \in F_q$. Writing this coordinatewise gives

$$\alpha_1(c_{1i}, c_{2i}, \dots, c_{5i}) + \alpha_2(c_{1j}, c_{2j}, \dots, c_{5j}) = 0.$$

Because of the structure of H_P, for any two distinct columns c_i and c_j, there exists a row index $k \in \{1,2,\dots,5\}$ such that exactly one of the entries c_{ki} or c_{kj} is non-zero.

Case 1: If $c_{ki} \neq 0$ and $c_{kj} = 0$, then from the k^{th} coordinate we obtain $\alpha_1 c_{ki} = 0$.

Since $c_{ki} \neq 0$, it follows that $\alpha_1 = 0$. As c_j is non-zero, we must also have $\alpha_2 = 0$.

Case 2: If $c_{ki} = 0$ and $c_{kj} \neq 0$, then similarly the k^{th} coordinate gives $\alpha_2 c_{kj} = 0$,

which implies $\alpha_2 = 0$, and hence $\alpha_1 = 0$.

Thus the equation $\alpha_1(c_i) + \alpha_2(c_j) = 0$

admits only the trivial solution. Therefore any two columns of H_P are linearly independent, and hence

$$d\left(C_q(G_P)\right) \geq 3.$$

Combining both inequalities, we conclude that

$$d\left(C_q(G_P)\right) = 3.$$

Hence the main parameters of the code generated by the G_P of the Petersen graph are $[15,10,3]_q$.

□

Since the Petersen graph is vertex-transitive and edge-transitive, its automorphism group can be used to construct permutation decoding sets for the associated code. Applying proposition 1, we obtain the following result.

Theorem 11. Let $C_q(G_P)$ be the linear code generated by the incidence matrix of the Petersen graph G_P. Since the Petersen graph is vertex and edge-transitive, any transitive subgroup of the automorphism group $Aut(G_P)$ of order 120 serves as a PD-set for full error correction of the code $C_q(G_P)$.

□

4.2 *Wagner Graph*

The Wagner graph, denoted by W, is a cubic graph on 8 vertices and 12 edges. It is also known as the Möbius ladder M_4. The graph is obtained from the cycle graph C_8 by joining opposite vertices with additional edges.

The vertex set of the Wagner graph is $V(W) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$.

The edge set is

$$E(W) = \{ v_1v_2,\ v_2v_3,\ v_3v_4,\ v_4v_5,\ v_5v_6,\ v_6v_7,\ v_7v_8,\ v_8v_1,\ \ v_1v_5,\ v_2v_6,\ v_3v_7,\ v_4v_8\}.$$

4.2.1 Construction of the Wagner Graph and the incidence matrix

The Wagner graph is constructed as follows:

1. Begin with the cycle graph $C_8 = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8)$.
2. Add the cycle edges v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_7v_8, v_8v_1.
3. Join opposite vertices of the cycle by adding the edges v_1v_5, v_2v_6, v_3v_7, v_4v_8.

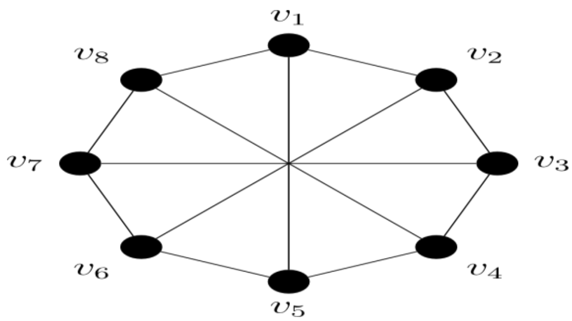

Figure 12: Wagner graph.

Then the incidence matrix G_W of the Wagner graph is

$$\left(\begin{array}{cccccccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1
\end{array}\right)$$

Figure 13: Generator matrix of Wagner graph.

Theorem 12. Let q be a prime integer with $q \neq 2$, and let $C_q(G_W)$ be the linear code over the finite field F_q generated by the G_W of the Wagner graph W. Then the code $C_q(G_W)$ has parameters

$$[12, 8, 3]_q..$$

Proof. The linear code $C_q(G_W)$ is constructed from the G_W of the Wagner graph W. Since the Wagner graph has 12 edges, the resulting code has length 12.

Furthermore, the Wagner graph is connected and non-bipartite. Therefore, by Proposition 2, the G_W over the finite field F_q, with $q \neq 2$, has full row rank equal to the number of vertices of the graph. Hence,

$$rank(G_W) = |V(W)| = 8.$$

Consequently, the dimension of the code $C_q(G_W)$ is 8.

To determine the minimum distance, by Proposition 3, it is enough to determine the smallest number of linearly dependent columns of H_W. A corresponding parity-check matrix H_W for the code is

$$\left(\begin{array}{cccccccc|cccc}
1 & q-1 & 0 & 0 & q-1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & q-1 & 0 & 0 & q-1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & q-1 & 0 & 0 & q-1 & 1 & 0 & 0 & 1 & 0 \\
q-1 & 0 & 0 & 1 & 1 & 0 & 0 & q-1 & 0 & 0 & 0 & 1
\end{array}\right).$$

Figure 14: Parity-check matrix of Wagner graph.

Let c_i denote the i^{th} column of H_W. We have $c_1 + c_2 + c_9 = (0,0,0,0)$, over the binary field, and more generally over F_q,

$$c_9 + (q-1)c_1 + (q-1)c_2 = (0,0,0,0).$$

Hence the three columns c_1, c_2, c_9 are linearly dependent. Therefore,

$$d\left(C_q(G_W)\right) \leq 3.$$

Now suppose that two columns c_i and c_j are linearly dependent. Then there exist $\alpha_1, \alpha_2 \in F_q$ such that $\alpha_1 c_i + \alpha_2 c_j = 0$.

Since any two distinct columns of H_W differ in at least one coordinate position where exactly one entry is non-zero, comparison of that coordinate immediately gives either $\alpha_1 = 0$ or $\alpha_2 = 0$.

Hence both coefficients must be zero, and therefore any two columns of H_W are linearly independent. Thus, $d\left(C_q(G_W)\right) \geq 3$.

Combining both inequalities yields $d\left(C_q(G_W)\right) = 3$.

Therefore the linear code generated by the incidence matrix of the Wagner graph has parameters $[12,8,3]_q$.

□

Theorem 13. Let $C_q(W)$ be the linear code generated by the incidence matrix of the Wagner graph W. Since the Wagner graph is vertex-transitive and edge-transitive, by proposition 1 any transitive subgroup of the automorphism group $Aut(G_W) \cong D_8$ of order 16 serves as a PD-set for full error correction of the code $C_q(G_W)$.

□

5 Conclusion

In this paper, we studied linear codes obtained from the incidence matrices of several bipartite and non-bipartite graphs over the finite field F_q. The main coding parameters, namely the length, dimension, and minimum distance, were determined for the associated graph-based codes. In particular, the Hypercube graph Q_n, Gray graph, Nauru graph, generalized Petersen graph $GP(5,2)$, and Wagner graph were considered, and the parameters of the corresponding linear codes were explicitly obtained. The results show that the structural and combinatorial properties of graphs have a strong influence on the behavior and characteristics of the resulting linear codes.

The work presented here also suggests a number of possible directions for future research. Similar techniques may be applied to other classes of regular graphs, cage graphs, and generalized Petersen graphs in order to construct new families of linear codes. Further study may also focus on additional properties of these codes, including automorphism groups, weight distributions, covering radius, and error-correcting performance. Another interesting direction is the search for optimal or near-optimal codes arising from highly symmetric and transitive graphs. Moreover, investigating properties such as self-orthogonality, self-duality, and efficient decoding procedures may lead to a deeper understanding of the relationship between graph theory and coding theory.

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